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ON THE STRUCTURE OF THE COHOMOLOGY OF NILPOTENT LIE ALGEBRAS

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Abstract

The exterior algebra over the centre of a Lie algebra acts on the cohomology of the Lie algebra in a natural way. Focusing on nilpotent Lie algebras, we explore the module structure afforded by this action. We show that for all two-step nilpotent Lie algebras, this module structure is non-trivial, which partially answers a conjecture of Cairns and Jessup [4]. The presence of free submodules indicates that the Lie algebra satisfies Halperin's Toral rank conjecture [11]. We prove that two specific classes of two-step nilpotent Lie algebras enjoy cohomology spaces with free submodules.

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Chapter 1

Introduction

The set of all possible states of a physical system can be considered to be a manifold M, sometimes equipped with additional structure, such as a metric or a symplectic form. If a Lie group G acts on the manifold M as a group of symmetries of this additional structure, then the data in the system "factors" into G and M/G, where M/G is the orbit space of the action. Furthermore, in many interesting cases, M/G is itself a manifold. This aids the understanding of the problem, since Lie groups are comparatively well-understood, and $\dim M/G = \dim M - \dim G$ (page 135, [8]). Therefore, the larger the group of symmetries, the smaller the dimension of M/G.

If a compact Lie group G acts almost freely on M, then the maximal torus of G also acts almost freely on M. This leads us to consider torus actions. Stephen Halperin conjectured an inequality which relates almost free torus actions with the dimension of the cohomology of M. Known as the toral rank conjecture (TRC), it was originally formulated in [11] and remains one of the intriguing unsolved problems of algebraic topology:

Conjecture 1.0.1. *Suppose an r-torus acts continuously with finite isotropy groups on a closed simply connected manifold M. Then,* $\dim H^*(M, \mathbb{R}) \geq 2^r$.

Informally, this conjecture states that an upper bound on the rank of a maximal torus acting almost freely on M is determined by the number of "holes" in M.

It has been shown to hold for certain classes of manifolds, such as compact connected Lie groups, homogeneous spaces G/H where G is a Lie group and H is a closed connected subgroup ([1], corollary 4.3.8 page 272), as well as Kähler manifolds ([1], remark 4.4.2 page 281).

One case that is still unsolved, and the one that interests us in particular is where the manifold in question is a *nilmanifold* N/D with N a nilpotent Lie group and D a co-compact discrete subgroup. The toral rank of N/D is related to the Lie algebra of N via the following key result.

Theorem 1.0.2 ([15]). *Let $\mathfrak{M} := N/D$ be a nilmanifold, where N is a nilpotent Lie group and D is a discrete subgroup. Then if L is the Lie algebra of N, we have:*

- $H^*(\mathfrak{M}) \cong H^*(L)$.
- *The dimension of a maximal torus acting freely on $\mathfrak{M}$ is equal to $\dim Z$, where Z is the centre of L.*

This allows us to formulate the toral rank conjecture for nilpotent Lie algebras:

Conjecture 1.0.3 (nil-TRC). *For any nilpotent Lie algebra L,*

$$\dim H^*(L) \geq 2^{\dim Z}$$

The nil-TRC is known for two-step nilpotent Lie algebras; a proof was first published in [5], and additional proofs have since appeared in [17] and [14]. However, the general case is still not known. It is worth noting that in many cases, the dimension bound is a considerable underestimate, one such instance is given by [17], another is provided in example 4.3.6.

One line of attack was proposed by Cairns and Jessup in [4]. They define a representation of ΛZ, the exterior algebra over the centre of a Lie algebra, as an algebra of operators in the cohomology of the Lie algebra. This defines a ΛZ-module structure on $H^*(L)$. This module structure is related to the toral rank conjecture by the following theorem, which for the sake of completeness, we prove in section 3.4.

Theorem 3.4.5([15]). *Let L be a Lie algebra, let Z be the centre of L, and let $0 \neq \tau \in \Lambda^{\dim Z} Z$. The following are equivalent:*

1. *There exists an element $\omega \in H^*(L)$ such that $i_\tau(\omega) \neq 0$.*

2. *The algebra homomorphism $i : \Lambda Z \to \mathrm{End}\,(H^*(L))$ is injective.*

3. *$H^*(L)$ contains a free ΛZ-submodule.*

If these conditions hold, then L satisfies the nil-TRC.

In sections 4.2, 4.3 and 4.4, we prove results about the ΛZ-module structure of three concrete classes of nilpotent Lie algebras.

In section 4.5 we prove the main result of this thesis:

Theorem 4.5.1 *Let $L := U \oplus Z$ be a two-step nilpotent Lie algebra with centre Z. Then the ΛZ-module structure of $H^*(L)$ is non-trivial.*

This is the two-step case of conjecture 5.9 of [4]:

Conjecture 3.4.6. *For any nilpotent Lie algebra L, $H^*(L)$ has a non-trivial ΛZ-module structure.*

Most proofs in chapters 2 and 3, as well as the first section of chapter 4, are well-known. The remainder of chapter 4 contains results which to our knowledge have not previously appeared in the literature.

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Chapter 2

Preliminaries

This chapter fixes notation for the rest of the thesis and collects a number of results from elementary multilinear algebra. Unless stated otherwise, all vector spaces and tensor products will be assumed to be over $\mathbb{R}$, and all vector spaces will be assumed to be finite dimensional, so in particular vector spaces are isomorphic to their duals. The dual $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of a vector space V will be denoted V^*.

2.1 Lie algebras

We start by defining graded Lie algebras. For the most part, we work only with classical Lie algebras, which are a special case of graded Lie algebras. A good reference for classical Lie algebra theory is [12]. Graded Lie algebras are studied in [6]. Our primary interest is in nilpotent and unimodular Lie algebras, and we restrict our attention to concepts important to these cases.

Definition 2.1.1. A *graded vector space* V is a direct sum of vector spaces:

$$V := V^0 \oplus V^1 \oplus \cdots \oplus V^n \oplus \cdots$$

If $x \in V^n$, we say x has *degree* n, and we let $|x| = n$.

Definition 2.1.2. A *graded Lie algebra* is a graded vector space A equipped with a bilinear map $[,] : A \times A \to A$, known as the graded Lie bracket, satisfying the following axioms:

- for $u \in A^p$ and $v \in A^q$, $[u, v] \in A^{p+q}$
- $[u, v] = -(-1)^{|u||v|}[v, u]$ (anti-symmetry)
- $[u, [v, w]] = [[u, v], w] + (-1)^{|u||v|}[v, [u, w]]$ (graded Jacobi identity)

Definition 2.1.3. A *Lie algebra* is a graded Lie algebra concentrated in degree 0.

Definition 2.1.4. If L is a Lie algebra and $A, B \subseteq L$, we define $[A, B]$ as the subspace spanned by all brackets $[a, b]$ with $a \in A$, $b \in B$. A vector subspace $I \subseteq L$ is an *ideal* if $[I, L] \subseteq I$. Note that this is equivalent to $[L, I] \subseteq I$, and unlike in non-commutative ring theory, Lie algebra ideals are always two-sided.

Definition 2.1.5. If $a, b \in L$ and $[a, b] = 0$, we say a and b *commute*. The centre Z of L is the set consisting elements which commute with all of L. Note that Z is an ideal of L and $[L, Z] = 0$.

Definition 2.1.6. The *adjoint representation* of a Lie algebra L is the linear map $\mathrm{ad} : L \to \mathrm{End}\, L$ defined by

$$\mathrm{ad}\, x(y) = [x, y].$$

A Lie algebra L is *unimodular* if for all $x \in L$, $\mathrm{tr}\, \mathrm{ad}\, x = 0$.

Let Z be the centre of L. It follows immediately from the definition of $\mathrm{ad}\, x$ and Z that $\mathrm{ad}\, x = 0$ if and only if $x \in Z$.

Definition 2.1.7. The *lower central series* of a Lie algebra L is the sequence of ideals defined inductively as follows:

- $L^1 = L$
- $L^k = [L, L^{k-1}]$

The Jacobi identity guarantees that each term of the above sequence is indeed an ideal, and $L^k \supseteq L^{k-1}$. If for some k, L^k is non-zero but L^{k+1} is zero, we say L is a *k-step nilpotent Lie algebra.*

The following proposition is crucial to our study of nilpotent Lie algebras.

Proposition 2.1.8. *Let L be a k-step nilpotent Lie algebra. Then,*

1. *L has a non-trivial centre.*

2. *L is unimodular.*

Proof. By definition, $L^k \neq 0$ and $[L, L^k] = 0$, so $L^k \subseteq Z$ where Z is the centre of L.

Now, fix $x \in L$. For all $y \in L$,

$$\underbrace{[x, [x, [x, \ldots [x, y] \ldots]]]}_{k \text{ times}} = 0$$

This is equivalent to

$$(\operatorname{ad} x)^k y = 0$$

showing that $\operatorname{ad} x$ is a nilpotent operator. However, every nilpotent operator has a strictly upper triangular matrix under a suitable basis, and so the sum of the diagonal elements is zero, showing that $\operatorname{tr} \operatorname{ad} x = 0$ for all x. Therefore L is unimodular. □

Remark 2.1.9. In the course of the above proof, it was shown that if L is nilpotent, then $\operatorname{ad} x$ is nilpotent for all x. The converse is also true; *Engel's theorem* is the statement of these two facts. We shall not need this result here.

Remark 2.1.10. The class of unimodular Lie algebras is much larger than the class of nilpotent Lie algebras. Section 3.2 gives an example of a unimodular Lie algebra which is not nilpotent.

The family of *Heisenberg Lie algebras* is a classical example of nilpotent Lie algebras.

Example 2.1.11. For every $n \in \mathbb{N}$, let $\mathfrak{h}_n := \mathrm{span}\{u_i | 1 \leq i \leq n\} \oplus \mathrm{span}\{v_i | 1 \leq i \leq n\} \oplus \mathrm{span}\{z\}$, with $[u_i, v_i] = z$ and all other brackets zero. Here $[\mathfrak{h}_n, \mathfrak{h}_n] = Z(\mathfrak{h}_n) = \mathrm{span}\{z\}$, so $\mathfrak{h}_n$ is a two-step nilpotent Lie algebra.

Example 2.1.12. A Lie algebra structure can be defined on any associative algebra, and in particular on any matrix algebra. This construction will be used in section 2.4.

Let A be an associative algebra over $\mathbb{R}$. Then we take L to be the underlying vector space of A with the Lie bracket given by $[x, y] = xy - yx$. Anti-symmetry is obvious and the Jacobi identity follows from the associativity of A.

If the centre of L is all of L, or equivalently all brackets are zero, we say that L is an *abelian* Lie algebra. In the previous example, the algebra A is abelian (commutative) if and only if the Lie algebra L is abelian.

2.2 Exterior algebras

All of the following results can be found in [7].

Definition 2.2.1. A *graded algebra* A is a graded vector space having an algebra structure where multiplication maps $A^a \otimes A^b$ into A^{a+b}, and $1 \in A^0$. If $A^0 \cong \mathbb{R}$, we say A is *connected*. In what follows, all graded algebras will be connected. If $x \in A^n$, we say that x is an element of *degree* n, and we let $|x| = n$.

Definition 2.2.2. Let V be a vector space. The free associative algebra on V is denoted $T(V)$ and defined as:

$$T(V) := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

Multiplication is defined as word concatenation on monomials, extended linearly:

$$(v_1 \otimes \cdots \otimes v_p) \cdot (w_1 \otimes \cdots \otimes w_q) = v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q$$

This is a graded algebra, and we write $T^p(V)$ to denote the subspace which is the p-fold tensor product of V with itself. Here $T^0(V) = \mathbb{R}$.

Remark 2.2.3. If A is a graded algebra and I is a graded ideal, then A/I is a graded algebra, since for all i, $I_i \subseteq A_i$, and there exists a vector space isomorphism:

$$A/I \cong \bigoplus_i A_i/I_i$$

Furthermore, the multiplication in A/I is induced by the multiplication in A, and the unit of A/I is $\overline{1} \in A_0/I_0$.

Definition 2.2.4. Let V be a vector space, and let $N \subseteq T(V)$ be the two-sided (graded) ideal generated by elements of the form $u \otimes v + v \otimes u$, for all $u, v \in V$. Then we define the *exterior algebra* over V by:

$$\Lambda V := T(V)/N$$

If we let $N^p := N \cap T^p(V)$, then the vector space $T^p(V)/N^p$ will be denoted $\Lambda^p V$, and we have:

$$\Lambda V = \bigoplus_{i \geq 0} \Lambda^i V$$

The product of $u, v \in \Lambda V$ will be denoted $u \wedge v$.

We now collect some standard results about the exterior algebra into a proposition.

Proposition 2.2.5. *If V is a vector space, the following holds for ΛV:*

1. *Since N is generated by elements of degree 2, we have $N^0 = N^1 = \{0\}$, so $\Lambda^0 V \cong \mathbb{R}$ and $\Lambda^1 V \cong V$.*

2. *The exterior algebra is graded-commutative – for elements $u \in \Lambda^p V$, $v \in \Lambda^q V$,*

$$u \wedge v = (-1)^{pq} v \wedge u$$

If $v \in \Lambda^p V$, we will write $|v| = p$.

3. *The dimension of* $\Lambda^p V$ *is* $\binom{\dim V}{p}$*, and so the dimension of* ΛV *is* $2^{\dim V}$*. In particular,* $\Lambda^m V = 0$ *if* $m > \dim V$*.*

Proof. Each part is proved as follows:

1. Immediate, since $T^0(V)/N^0 = T^0(V) \cong \mathbb{R}$, and $T^0(V)/N^0 = T^1(V) \cong V$.

2. It suffices to prove this fact for monomials $u \in \Lambda^p V$, $v \in \Lambda^q V$.

   We write u and v as products of generators:

$$\begin{aligned} u &= u_1 \wedge \cdots \wedge u_p \\ v &= v_1 \wedge \cdots \wedge v_q \end{aligned}$$

   From the definition of N, we can conclude that:

$$u_1 \wedge \cdots \wedge u_p \wedge v_1 = -u_1 \wedge \cdots \wedge u_{p-1} \wedge v_1 \wedge u_p$$

   By induction on p, we get:

$$u_1 \wedge \cdots \wedge u_p \wedge v_1 = (-1)^p v_1 \wedge u_1 \wedge \cdots \wedge u_p$$

   Now we write

$$u \wedge v = u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q$$

   and proceed to shift the v_i to the left, starting with v_1. By induction on q, we get:

$$u \wedge v = (-1)^{pq} v \wedge u$$

3. Let $n := \dim V$. By (2), the vector space $\Lambda^p V$ is spanned by all non-zero monomials $x_{i_1} \wedge \cdots \wedge x_{i_p}$, where $\{x_i | 1 \leq i \leq n\}$ is a basis of V. Such a monomial is non-zero precisely when the i_j's are distinct, and different permutations of the i_j's only change the sign of the resulting monomial. We also remark that these monomials are linearly independent. So the number of distinct monomials is the number of distinct p-element subsets of a set of n elements, which is $\binom{n}{p}$.

The second part follows from the binomial identity:

$$\sum_k \binom{n}{k} = 2^n.$$

□

The exterior algebra satisfies a universal property.

Theorem 2.2.6. *Let $\phi : V \times V \to W$ be an anti-symmetric bilinear map. Then there exists a unique linear map $\overline{\phi} : \Lambda^2 V \to W$ such that $\overline{\phi}(u \wedge v) = \phi(u, v)$.*

Proof. By the universal property of tensor products, ϕ induces a unique linear map $\tilde{\phi} : V \otimes V \to W$. The ideal N defined previously is contained in $\ker \tilde{\phi}$ since ϕ is anti-symmetric, so the map $\overline{\phi}(u \wedge v) := \tilde{\phi}(u \otimes v)$ is well-defined.

It remains to show that $\overline{\phi}$ is unique. If $\phi' : \Lambda^2 V \to W$ is another map satisfying $\phi'(u \wedge v) = \phi(u, v)$, then $(\overline{\phi} - \phi')(u \wedge v) = 0$, and since vectors of the form $u \wedge v$ span $\Lambda^2 V$, we see that $\overline{\phi} = \phi'$. □

Definition 2.2.7. The *dual pairing* of $\Lambda^m V^*$ on $\Lambda^n V$ is defined as follows.

If $m = n$, for monomials $v_1 \wedge \cdots \wedge v_m$ and $w_1 \wedge \cdots \wedge w_m$, we set

$$\langle v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_m \rangle = \det[v_i(w_j)]_{1 \leq i,j \leq m}.$$

This is well-defined and bilinear by the elementary properties of the determinant.

If $m \neq n$,

$$\langle \Lambda^m V^*, \Lambda^n V \rangle = 0.$$

Proposition 2.2.8. *If V is a vector space with dual V^*, there exists a natural vector space isomorphism:*

$$(\Lambda V)^* \cong \Lambda(V^*)$$

Proof. First we show that the dual pairing of ΛV^* on ΛV is non-degenerate. Let $0 \neq w_1 \wedge \cdots w_m \in \Lambda V$ be given. View $\{w_i\}_{1 \leq i \leq m}$ as a basis of a subspace of V and consider a dual basis $\{v_i\}_{1\leq i\leq m}$ which satisfies the following:

$$v_i(w_j) = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j \end{cases}$$

Since $\{v_i\}_{1\leq i\leq m}$ is linearly independent, we have $v_1 \wedge \cdots \wedge v_m \neq 0$. We compute

$$\langle v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots w_m\rangle = \det[v_i(w_j)]_{1\leq i,j\leq m} = \det 1_m = 1$$

This shows $\langle , \rangle$ is non-degenerate. Now, we define an isomorphism

$$\phi : \Lambda(V^*) \to (\Lambda V)^*$$

by

$$\phi(\alpha)(\beta) := \langle \alpha, \beta\rangle.$$

The non-degeneracy of $\langle , \rangle$ yields that ϕ is an isomorphism, and since no choice of basis was made, ϕ is natural. □

2.3 Derivations

In this section, we study an important class of operators sitting inside $\mathrm{End}\,(\Lambda V)$.

Definition 2.3.1. A linear map $\phi : \Lambda V \to \Lambda V$ has degree k if it maps $\Lambda^p V$ into $\Lambda^{p+k} V$. We say that ϕ is a *derivation* of degree k if it also satisfies the *graded Leibnitz rule*:

$$\phi(u \wedge v) = \phi(u) \wedge v + (-1)^{k|u|} u \wedge \phi(v)$$

A derivation d which also satisfies $d^2 = 0$ is called a *differential*.

Remark 2.3.2. A derivation $\phi : \Lambda V \to \Lambda V$ is always zero on $\Lambda^0 V \cong \mathbb{R}$.

Indeed,

$$\begin{aligned} \phi(1) &= \phi(1 \cdot 1) \\ &= \phi(1) \cdot 1 + (-1)^{k\cdot|1|} 1 \cdot \phi(1) \\ &= \phi(1) + \phi(1), \end{aligned}$$

so $\phi(1) = 0$.

Proposition 2.3.3. *Let $d : \Lambda V \to \Lambda V$ be a differential of degree k. The kernel of d is a graded subalgebra of ΛV, and the image of d is a graded ideal in* $\ker d$.

Proof. Since d is a linear map of degree k, $\ker d$ and $\operatorname{im} d$ are both graded.

Suppose $\alpha, \beta \in \ker d$. Then,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k|\alpha|}\alpha \wedge d\beta = 0$$

Furthermore, by remark 2.3.2, $1 \in \ker d$. Thus the kernel is a subalgebra.

To see that the image is an ideal of the kernel, first note that $d^2 = 0$ is equivalent to $\operatorname{im} d \subseteq \ker d$. Now, pick $d\alpha \in \operatorname{im} d$, $\beta \in \ker d$. We have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k|\alpha|}\alpha \wedge d\beta = d\alpha \wedge \beta$$

Hence $d\alpha \wedge \beta \in \operatorname{im} d$. □

Definition 2.3.4. A *graded-commutative algebra* is a graded algebra A such that for $a, b \in A$,

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

A *graded-commutative differential algebra* is a pair (A, d) where A is a graded algebra and d is a differential. The graded algebra $\ker d / \operatorname{im} d$ is called the *cohomology* of A and is denoted by $H^*(A, d)$, or if the differential is apparent from context, $H^*(A)$.

Proposition 2.3.5. *Let (A, d) and (B, D) be graded-commutative differential algebras. If $\phi : A \to B$ is a linear map of degree k which satisfies*

$$\phi d - (-1)^k D\phi = 0$$

then ϕ induces a well-defined map

$$H^*(\phi) : H^*(A, d) \to H^*(B, D).$$

Proof. Let $x \in \ker d$, that is $dx = 0$. We know $0 = \phi dx = (-1)^k D\phi(x)$, hence $\phi(x) \in \ker D$. Now choose $dy \in \operatorname{im} d$. We have $\phi dy = (-1)^k D\phi y \in \operatorname{im} D$. So ϕ restricts to a map $\phi : \ker d \to \ker D$, and factors to a map $H^*(\phi) : \ker d/\operatorname{im} d \to \ker D/\operatorname{im} D$. □

In the language of category theory[1], H^* is a *functor* from the category of graded-commutative differential algebras to the category of graded-commutative algebras.

Having demonstrated some key properties possessed by differentials, we return to arbitrary derivations.

Proposition 2.3.6. *Let $\phi : \Lambda V \to \Lambda V$ be a derivation of degree k. Then ϕ is completely determined by its action on V.*

Proof. Suppose $\phi : \Lambda V \to \Lambda V$ and $\theta : \Lambda V \to \Lambda V$ are both derivations of degree k which are equal on V. We must show $\phi = \theta$. The proof proceeds by induction; we show $\phi - \theta$ is zero on $\Lambda^p V$ for all p.

The case $p = 0$ is true by remark 2.3.2 and $p = 1$ is true by assumption. For the inductive step, assume ϕ and θ agree on $\Lambda^i V$ for $i < p$. Pick a monomial in $\Lambda^p V$ and write it as $x \wedge y$, where $|x| = 1$ and $|y| = p - 1$. We now compute $\phi - \theta$ on $x \wedge y$:

$$(\phi - \theta)(x \wedge y) = (\phi - \theta)(x) \wedge y + (-1)^{k \cdot |x|} x \wedge (\phi - \theta)(y).$$

Since $x \in \Lambda^1 V$, $(\phi - \theta)(x) = 0$, and since $y \in \Lambda^{p-1} V$, $(\phi - \theta)(y) = 0$ by the induction hypothesis. Therefore on $\Lambda^p V$, $\phi = \theta$, and by induction this holds on all of ΛV. □

Two remarks follow immediately.

Remark 2.3.7. Any linear map $\phi : V \to \Lambda^k V$ can be inductively extended to a *unique* derivation of degree k. Define $\phi(1) := 0$; now given $u \wedge v \in \Lambda^p V$ where $|u| = 1$, we set:

$$\phi(u \wedge v) := \phi(u) \wedge v + (-1)^k u \wedge \phi(v)$$

Since $|v| = p - 1$ we apply the definition inductively to v.

[1] See, for example, [13].

Remark 2.3.8. Derivations of degree $k < -1$ are identically zero, since a derivation of degree $k < -1$ is determined by its action on V, yet it must map V to $\Lambda^{1-k}V$, which is zero since $1 - k < 0$.

Definition 2.3.9. Let $\mathrm{Der}\,(\Lambda V)^k$ be the space of all derivations of degree k on ΛV. Define a graded vector space

$$\mathrm{Der}\,(\Lambda V) = \bigoplus_k \mathrm{Der}\,(\Lambda V)^k$$

In fact, $\mathrm{Der}\,(\Lambda V)$ has the structure of a graded Lie algebra.

Theorem 2.3.10. *For any vector space V, $\mathrm{Der}\,(\Lambda V)$ is a graded Lie algebra, with bracket defined by*

$$[\phi, \theta](x) = \phi(\theta(x)) - (-1)^{|\phi||\theta|}\theta(\phi(x))$$

where $x \in \Lambda V$ is arbitrary.

Proof. If $|\phi| = k$ and $|\theta| = l$, it is clear that $[\phi, \theta]$ is a linear map of degree $k + l$. A tedious but straightforward calculation shows that $[\phi, \theta] \in \mathrm{Der}\,(\Lambda V)$. Anti-symmetry is clear and the Jacobi identity follows from the associativity of function composition. ☐

Now we turn our attention to a very specific class of derivations.

Definition 2.3.11. (Interior product) For $z \in V$, the map $i_z : \Lambda V^* \to \Lambda V^*$ is the transpose of multiplication on the left by z on ΛV, under the dual pairing of ΛV^* and ΛV; for any $u \in \Lambda V^{n+1}$ and $v \in \Lambda V^n$,

$$\langle i_z(u), v\rangle = \langle u, z \wedge v\rangle$$

Lemma 2.3.12. *The map i_z is a derivation of degree -1.*

Proof. First, we note that for $u \in V^*$,

$$i_z(u) = \langle i_z(u), 1\rangle = \langle u, z \wedge 1\rangle = \langle u, z\rangle.$$

Let $\{u_i\}_{1\leq i\leq p} \subset V^*$ and $\{v_i\}_{1\leq i\leq p-1} \subset V$ be two lists of vectors, possibly linearly dependent. We compute:

$$\begin{aligned}\langle i_z(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_{p-1}\rangle &= \langle u_1 \wedge \cdots \wedge u_p, z \wedge v_1 \wedge \cdots \wedge v_{p-1}\rangle \\ &= \begin{vmatrix} \langle u_1, z\rangle & \langle u_1, v_1\rangle & \cdots & \langle u_1, v_{p-1}\rangle \\ \langle u_2, z\rangle & \langle u_2, v_1\rangle & \cdots & \langle u_2, v_{p-1}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, z\rangle & \langle u_p, v_1\rangle & \cdots & \langle u_p, v_{p-1}\rangle \end{vmatrix}\end{aligned}$$

Call this matrix M. We expand the determinant of M along the first column; using M_{ij} to denote the determinant of M with the ith row and jth column deleted,

$$\begin{aligned}\det M &= \sum_j (-1)^{j+1}\langle u_j, z\rangle \det M_{j1} \\ &= \sum_j (-1)^{j+1}\langle u_j, z\rangle\langle u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_{p-1}\rangle \\ &= \sum_j (-1)^{j+1}\langle i_z(u_j) \wedge u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_{p-1}\rangle \\ &= \langle \sum_j (-1)^{j+1} i_z(u_j) \wedge u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_{p-1}\rangle\end{aligned}$$

Hence,

$$i_z(u_1 \wedge \cdots \wedge u_p) = \sum_j (-1)^{j+1} i_z(u_j) \wedge u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_p$$

This shows that i_z is a derivation of degree -1 on monomials. The general result now follows by linearity. □

These maps i_z are used to define *cohomology operations* in section 3.4.

Remark 2.3.13. For $\zeta \in \Lambda V$ with $|\zeta| > 1$, we can define a map i_ζ as above; however in this case it fails to be a derivation. Indeed, were it to be a derivation of degree $-|\zeta| < -1$, remark 2.3.8 would force it to be identically zero, yet for non-zero ζ of any degree, there exists $\zeta^* \in \Lambda V^*$ such that $\langle i_\zeta(\zeta^*), 1\rangle = \langle \zeta^*, \zeta\rangle \neq 0$.

2.4 Skew-adjoint operators

We recall some elementary results from the theory of linear operators, the full details of which can be found in many texts, for example [3].

If V is a (real or Hermitian) inner product space, then a linear operator $T : V \to V$ is *self-adjoint* if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$. Similarly, T is *skew-adjoint* if $\langle Tv, w\rangle = -\langle v, Tw\rangle$.

Definition 2.4.1. The space of skew-adjoint linear operators is denoted $\mathfrak{so}(V)$. It is easy to see that it has the structure of a Lie algebra[2], with $[T, S] = TS - ST$.

A standard result from linear algebra states that a self-adjoint operator is diagonalizable. Here we present the proof of the well-known canonical form for skew-adjoint operators.

Proposition 2.4.2. *Let V be an inner product space. If $T \in \mathfrak{so}(V)$, then there exists a basis of V such that the matrix of T with respect to this basis is block diagonal of the following form, where $\lambda_i \in \mathbb{R}$:*

$$T = \begin{bmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{bmatrix}$$

Proof. Recall that given a real inner product space V, we construct the *scalar extension* $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$, and equip $V_{\mathbb{C}}$ with a Hermitian inner product as follows, where $u, v \in V$ and $a, b \in \mathbb{C}$:

$$\langle u \otimes a, v \otimes b\rangle = a\bar{b}\langle u, v\rangle.$$

[2] This is the Lie algebra obtained from the Lie group $SO(V)$ of special orthogonal linear operators on V.

We will henceforth write $u \otimes a$ as ua. The following identity is satisfied:

$$\langle ua, vb\rangle = \overline{\langle vb, ua\rangle}$$

Extend T to an operator acting on $V_{\mathbb{C}}$ by $T(ua) = aT(u)$, so that $iT(ua) := iaTu$ is self-adjoint:

$$\langle iTv, w\rangle = \langle Tv, -iw\rangle = \langle v, iTw\rangle$$

Since iT is self-adjoint, we can find a basis $\{x_i\}$ of $V_{\mathbb{C}}$ such that iT is a diagonal matrix with real eigenvalues on the diagonal. Therefore, $T = -i \cdot i \cdot T$ can be written as follows:

$$T = \begin{bmatrix} -i\lambda_1 & & \\ & -i\lambda_2 & \\ & & \ddots \end{bmatrix}$$

If λ is an eigenvalue of iT, we denote the corresponding eigenspace by E_λ. We may write a general element of E_{λ_j} in the form $v + iw$, $v, w \in V$. We see that $T(v+iw) = -i\lambda_j(v+iw) = -i\lambda_j v+\lambda_j w$, and on the other hand, $T(v+iw) = Tv+iTw$. Since $V_{\mathbb{C}} \cong V \oplus iV$, equating real and imaginary parts yields:

$$Tv = \lambda_j w \tag{2.4.1}$$

$$Tw = -\lambda_j v \tag{2.4.2}$$

The vectors v and w are either linearly dependent or independent. We treat the two cases separately:

- **v and w are linearly dependent.** Either at least one is zero, or $v = aw$ for some $a \in \mathbb{R}$. If at least one is zero, then either $\lambda_j = 0$, or 2.4.1 shows that both are necessarily zero, which contradicts that $v + iw$ is an eigenvector. The other possibility is that v and w are both non-zero and $v = aw$, in which case $Tv = aTw = -a^2\lambda_j w$. But we have $Tv = \lambda_j w$, so either $\lambda_j = 0$, or $-a^2 = 1$. The latter is impossible since $a \in \mathbb{R}$. So $\lambda_j = 0$ and $T|_{\text{span}\{v,w\}} = 0$.

- v **and** w **are linearly independent.** Let $F_j := \text{span}\{w, v\}$. Then $T|_{F_{\lambda_j}}$ has the following matrix:

$$\begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}$$

This completes the proof. □

Now we come to the main result of this section.

Proposition 2.4.3. *Let $T, S \in \mathfrak{so}(V)$. If $[T, S] = 0$, then there exists a basis making the matrices of both T and S block diagonal in the sense of proposition 2.4.2.*

Proof. If λ is an eigenvalue of iT, we denote the corresponding eigenspace as E_λ. If $0 \neq v \in E_\lambda$, then $iTSv = iSTv$, thus $iTSv = \lambda Sv$ and $Sv \in E_\lambda$. Therefore we can write V as a direct sum of iS-invariant eigenspaces of iT. For each eigenvalue λ, $iT|_{E_\lambda}$ is multiplication by a scalar, and since $iS|_{E_\lambda}$ is self-adjoint, we can find a basis for each E_λ making $iS|_{E_\lambda}$ diagonal. Since iT is multiplication by a scalar on every E_λ, it remains diagonal under the new basis, so we have a basis where both operators are diagonal in the usual sense. The desired forms of T and S on V now follow. □

2.5 The structure of $\Lambda^2 V$

In this section we focus on degree 2 elements of an exterior algebra. The results will be fundamental to the study of the structure of two-step nilpotent Lie algebras in section 4.3.

Theorem 2.5.1. *For any real inner product space V, there exists a vector space isomorphism:*

$$\eta : \mathfrak{so}(V) \to \Lambda^2 V^*$$

Proof. Given $T \in \mathfrak{so}(V)$, Define a bilinear form by $\overline{T}(u, v) = \langle Tu, v\rangle$. Since T is skew-adjoint, $\overline{T}$ is anti-symmetric, therefore by theorem 2.2.6 it induces a linear map $T' : \Lambda^2 V \to \mathbb{R}$. Hence $T' \in \Lambda^2 V^*$. Set $\eta(T) := T'$.

Linearity of η follows from the universal property.

To see that η is injective, suppose that $\eta(T) = 0$ for some T. Then, $T' : \Lambda^2 V \to \mathbb{R}$ is the zero map, thus $T : V \times V \to \mathbb{R}$ is the zero form, i.e. for all $u, v \in V$, $\langle Tu, v\rangle = 0$. This implies that $Tu = 0$ for all $u \in V$, and so $T = 0$.

On the other hand, a dimension argument shows that η is surjective. Let $n := \dim V$. A skew-adjoint linear map is completely determined by the upper triangle of its matrix, and thus we get:

$$\dim \mathfrak{so}(V) = \frac{n(n-1)}{2} = \binom{n}{2} = \dim \Lambda^2 V = \dim \Lambda^2 V^*$$

It now follows that η is an isomorphism. $\square$

Corollary 2.5.2 (Easy Darboux Theorem)**.** *Let* $\omega \in \Lambda^2 V^*$. *Then there exists a* symplectic basis *of* V^*, $\{v_i\}_{1\le i\le \dim V^*}$, *such that:*

$$\omega = v_1 \wedge v_2 + \cdots + v_{2j-1} \wedge v_{2j}.$$

The integer j *is the* rank *of* ω, *denoted* $\operatorname{rank}\omega$.

Proof. Recall η, the vector space isomorphism defined in theorem 2.5.1. We can find a basis of V^* such that $\eta^{-1}(\omega)$ is block diagonal:

$$\eta^{-1}(\omega) = \begin{bmatrix} 0 & \lambda_1 & \\ -\lambda_1 & 0 & \\ & & \ddots \end{bmatrix}$$

Now our result follows from the definition of η:

$$\omega = \sum_{1\le i\le \dim V^*} -\lambda_i e_{2i-1} \wedge e_{2i}$$

$\square$

We can define a Lie algebra structure on $\Lambda^2 V^*$. For $\alpha, \beta \in \Lambda^2 V^*$,

$$[\alpha, \beta] := \eta([\eta^{-1}(\alpha), \eta^{-1}(\beta)]).$$

Corollary 2.5.3. *Two elements $\alpha, \beta \in \Lambda^2 V^*$ commute if and only if there exists a basis $\{x_i\}_{1\leq i\leq n}$ of V^* such that α and β have the following form, for scalars $a_j, b_j \in \mathbb{R}$:*

$$\begin{aligned} \alpha &= \sum_{1\leq j\leq n} a_j x_{2j-1} \wedge x_{2j} \\ \beta &= \sum_{1\leq j\leq n} b_j x_{2j-1} \wedge x_{2j}. \end{aligned}$$

Proof. Note that $[\alpha, \beta] = 0$ if and only if $\eta^{-1}(\alpha)$ and $\eta^{-1}(\beta)$ are simultaneously diagonalizable in the sense of lemma 2.4.3, and by corollary 2.5.2, this holds if and only if there is a basis such that α and β have the desired form. ☐

2.6 Some graph theory

To be self-contained, we mention some basic facts about graph theory which will be used in the proof of theorem 4.4.1. Full details can be found in any text on elementary graph theory, for example [10].

Definition 2.6.1. A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a finite set of *vertices* and $E(G)$ is a finite list of *edges*. An edge is an unordered pair v_1, v_2 where $v_1, v_2 \in V(G)$. There may be multiple edges between the same pair of vertices. (In some texts, this is known as a *multi-graph*.)

Definition 2.6.2. Let G be a graph, and let $v \in V(G)$. The *degree* of v is the cardinality of the set

$$\{e \in E(G) | v \in e\}$$

Definition 2.6.3. Let G_1 and G_2 be two graphs. The *union* $G_1 \cup G_2$ is defined to be the graph with vertex set $V(G_1) \cup V(G_2)$, and the edge list being the concatenation (or disjoint union) of the lists $E(G_1)$ and $E(G_2)$.

Definition 2.6.4. A *cycle* in a graph G is a list of vertices satisfying three conditions:

- from each of its vertices there is an edge to the next vertex,

- there are no repeated vertices,
- there is an edge from the last vertex to the first vertex.

The *length* of a cycle is number of vertices in the cycle.

Definition 2.6.5. A graph G is said to be *n-regular* if every vertex has degree n.

We prove a well-known fact about 2-regular graphs.

Lemma 2.6.6. *A 2-regular graph is a disjoint union of cycles.*

Proof. Starting from an initial vertex v_0, we form a cycle as follows:

- We pick one of the two edges leaving v_0, and call the vertex opposite that edge v_1.
- Now there are two edges leaving v_1, one of which was already traversed, giving us only one way to proceed. We label the vertex opposite that edge v_2, and continue the process.
- Since $V(G)$ is finite, eventually the process must return us to v_0, forming a cycle.

If possible, we then choose a vertex not part of the cycle as v_0, and repeat the above steps. Eventually this algorithm will terminate, as every vertex will be part of some cycle. The cycles will be disjoint; if a vertex appears in a cycle, then both edges leaving that vertex appear in the cycle, but G is 2-regular, so a vertex can only appear in a single cycle. ☐

Definition 2.6.7. Let G be a graph, and let $[n]$ be the set $\{1, 2, \ldots, n\}$. An *n-colouring* of G is a map $f : V(G) \to [n]$ such that no two vertices sharing an edge are mapped to the same element by f.

Now we can prove a lemma we will need in section 4.4.

Lemma 2.6.8. *Let G_1 and G_2 be 1-regular graphs sharing the same vertex set, that is $V(G_1) = V(G_2)$. Then $G := G_1 \cup G_2$ is a 2-regular graph which admits 2^c distinct 2-colourings, where c is the number of cycles in G. Furthermore, each 2-colouring partitions the set of vertices into two equal-sized subsets.*

Proof. Since $E(G) = E(G_1) \cup E(G_2)$, it is clear that there are two edges leaving every vertex.

Lemma 2.6.6 implies that G is a union of disjoint cycles. Now, suppose v is an arbitrary vertex part of some cycle. Since G is a union of two 1-regular graphs, one of the edges leaving v must come from G_1, and the other edge must come from G_2. As we traverse the cycle from start to finish, we alternate between edges from G_1 and edges from G_2. An even number of edges must be traversed, since we return to the original vertex. Therefore, each cycle has even length.

Note that a cycle of even length can be 2-coloured in one of two ways; once the colour of a single vertex has been fixed, the colours of all remaining vertices are completely determined. There are two choices one can make for the colour of the initial vertex. If G is a union of c even-length cycles, then it admits 2^c distinct 2-colourings.

Note that when a 2-colouring of a cycle is chosen, each distinct colour is assigned to an equal number of vertices. From this it follows that in G, each distinct colour is also assigned to an equal number of vertices. □

Chapter 3

Lie algebra cohomology and ΛZ-modules

We begin by recalling the definition of $H^*(L)$, the Lie algebra cohomology of L. Section 3.4 then defines a ΛZ-module structure on $H^*(L)$ as introduced in [4].

3.1 Lie algebra cohomology

The cohomology of a Lie algebra L is defined as the cohomology of $(\Lambda L^*, d)$, where the differential d is determined by the Lie bracket on L. The algebra $(\Lambda L^*, d)$ is called the *Koszul complex* of L.

Theorem 2.2.6 implies that the Lie bracket induces a well-defined linear map $\partial : \Lambda^2 L \to L$. The transpose of this map under the dual pairing of ΛL^* with ΛL will be denoted d. By the definition of the transpose, for $z \in L^*$ and $x, y \in L$, we have

$$\langle dz, x \wedge y \rangle = \langle z, [x, y] \rangle.$$

The map $d : L^* \to \Lambda^2 L^*$ can be extended to a unique derivation $d : \Lambda L^* \to \Lambda L^*$ of degree 1 as in proposition 2.3.6.

Theorem 3.1.1 ([14], page 6, or [9], page 175)**.** *The map d is a differential: that is, $d^2 = 0$.*

Proof. Since $\mathrm{Der}\,(\Lambda V^*)$ is a graded Lie algebra and d has degree 1, $[d,d] = 2d^2$, so d^2 is also a derivation. Now, proposition 2.3.6 ensures that d^2 is determined by its action on L^*, so we are done if we can show that $d^2 : L^* \to \Lambda^3 L^*$ is zero.

Define $\Phi : \Lambda^3 L \to \Lambda^2 L$ as follows:

$$\Phi(x \wedge y \wedge z) = [x,y] \wedge z + [z,x] \wedge y + [y,z] \wedge x$$

Now by the Jacobi identity,

$$\partial(\Phi(x \wedge y \wedge z)) = [[x,y],z] + [[z,x],y] + [[y,z],x] = 0$$

Hence $\partial \circ \Phi = 0$.

It remains to show that Φ is the transpose of $d : \Lambda^2 L^* \to \Lambda^3 L^*$, since then $d^2 = d \circ d = (\partial \circ \Phi)^* = 0$.

Let $a, b \in L^*$ and $x, y, z \in L$. Then we have:

$$\langle \Phi^*(a \wedge b), x \wedge y \wedge z \rangle = \langle a \wedge b, \Phi(x \wedge y \wedge z) \rangle \tag{3.1.1}$$

Now:

$$\begin{aligned}
\langle a \wedge b, \partial(x \wedge y) \wedge z \rangle &= -\langle a \wedge b, z \wedge \partial(x \wedge y) \rangle \\
&= -\langle i_z(a \wedge b), \partial(x \wedge y) \rangle \\
&= -\langle a, z \rangle \langle b, \partial(x \wedge y) \rangle + \langle b, z \rangle \langle a, \partial(x \wedge y) \rangle \\
&= -\langle a, z \rangle \langle db, x \wedge y \rangle + \langle b, z \rangle \langle da, x \wedge y \rangle
\end{aligned}$$

Similar computations yield:

$$\begin{aligned}
\langle a \wedge b, \partial(z \wedge x) \wedge y \rangle &= -\langle a, y \rangle \langle db, z \wedge x \rangle + \langle b, y \rangle \langle da, z \wedge x \rangle \\
\langle a \wedge b, \partial(y \wedge z) \wedge x \rangle &= -\langle a, x \rangle \langle db, y \wedge x \rangle + \langle b, x \rangle \langle da, y \wedge z \rangle
\end{aligned}$$

Substituting everything into equation 3.1.1 gives us the following:

$$\begin{aligned}\langle \Phi^*(a \wedge b), x \wedge y \wedge z\rangle &= -\langle a, z\rangle\langle db, x \wedge y\rangle + \langle b, z\rangle\langle da, x \wedge y\rangle \\ &\quad -\langle a, y\rangle\langle db, z \wedge x\rangle + \langle b, y\rangle\langle da, z \wedge x\rangle \\ &\quad -\langle a, x\rangle\langle db, y \wedge x\rangle + \langle b, x\rangle\langle da, y \wedge z\rangle \\ &= \langle da, i_b(x \wedge y \wedge z)\rangle - \langle db, i_a(x \wedge y \wedge z)\rangle \\ &= \langle (da) \wedge b - a \wedge (db), x \wedge y \wedge z\rangle \\ &= \langle d(a \wedge b), x \wedge y \wedge z\rangle\end{aligned}$$

Thus $\Phi^* = d$ on $\Lambda^3 L^*$, and since $\partial \circ \Phi = 0$ it follows that $d^2 = 0$. □

We are now ready to define the central concept studied in this thesis.

Definition 3.1.2 (Lie algebra cohomology)**.** Let L be a Lie algebra. We define a differential d on ΛL^* as above, and let $H^*(L) := \ker d / \operatorname{im} d$. This is a graded-commutative algebra by lemma 2.3.3, and is known as the *cohomology* of L. The dimension of $H^n(L)$ is the nth *Betti number* of L.

The following useful lemma provides a connection between the adjoint representation of L, and the Koszul complex of L.

Lemma 3.1.3. *For $x \in L$ and $y \in L^*$,*

$$(\operatorname{ad} x)^*(y) = i_x dy$$

Proof. We show that $i_x d$ is the transpose of $\operatorname{ad} x$; since transposes are unique, the result follows. Indeed, for $w \in L$,

$$\begin{aligned}\langle i_x dy, w\rangle &= \langle dy, x \wedge w\rangle \\ &= \langle y, [x, w]\rangle \\ &= \langle y, \operatorname{ad} x(w)\rangle.\end{aligned}$$

□

3.2 Example cohomology computations

This section contains a few examples of explicit cohomology computations.

Example 3.2.1. The cross product on $\mathbb{R}^3 = \text{span}\{x, y, z\}$ is a skew-symmetric bilinear map defined on basis elements:

$$\begin{aligned} x \times y &:= z \\ y \times z &:= x \\ z \times x &:= y \end{aligned}$$

This makes $\mathbb{R}^3$ into a Lie algebra, which we shall call L. Note that L is unimodular but not nilpotent. In the literature, this Lie algebra is denoted $\mathfrak{so}(3)$. It can also be presented as a matrix Lie algebra without reference to the cross product.

Fixing a dual basis, we have $L^* = \text{span}\{x^*, y^*, z^*\}$. The differential on L^* is given by:

$$\begin{aligned} dx^* &= y^* \wedge z^* \\ dy^* &= z^* \wedge x^* \\ dz^* &= x^* \wedge y^* \end{aligned}$$

Note that there are no non-trivial cycles in L^*. Recall that $\dim \Lambda^2 L^* = \binom{3}{2} = 3$, so $\Lambda^2 L^*$ is spanned by boundaries, and $\dim \Lambda^3 L^* = \binom{3}{3} = 1$, so $\Lambda^3 L^*$ is spanned by a single element $x^* \wedge y^* \wedge z^*$, which is a cycle for degree reasons. Hence, we have:

$$\begin{aligned} H^0(L) &= \text{span}\{1\} \cong \mathbb{R} \\ H^1(L) &= 0 \\ H^2(L) &= 0 \\ H^3(L) &= \text{span}\{x^* \wedge y^* \wedge z^*\} \end{aligned}$$

Remark 3.2.2. For any Lie algebra L, $H^0(L) \cong \mathbb{R}$, since $1 \in \ker d$, and $1 \notin \operatorname{im} d$. Computations similar to those in the example above show that for any Lie algebra L, $H^1(L) \cong L/[L, L]$.

Example 3.2.3. Let $L := \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$, with non-zero brackets on basis elements defined by

$$\begin{aligned} [x_1, x_2] &:= z_1 \\ [x_3, x_4] &:= z_1 \\ [x_1, x_3] &:= z_2 \\ [x_2, x_4] &:= z_2. \end{aligned}$$

Under the corresponding dual basis, the differential is defined as follows:

$$\begin{aligned} dz_1^* &= x_1^* \wedge x_2^* + x_3^* \wedge x_4^* \\ dz_2^* &= x_1^* \wedge x_3^* + x_2^* \wedge x_4^* \end{aligned}$$

The Koszul complex $(\Lambda L^*, d)$ is a 64-dimensional vector space; computer calculations produce the following basis for $H^*(L)$:

- $H^1(L)$:

  x_1^*

  x_2^*

  x_3^*

  x_4^*

- $H^2(L)$:

  $x_2^* \wedge x_3^*$

  $x_1^* \wedge x_4^*$

  $x_2^* \wedge x_4^* - x_1^* \wedge x_3^*$

  $x_3^* \wedge x_4^* - x_1^* \wedge x_2^*$

  $-x_4^* \wedge z_1^* + x_1^* \wedge z_2^*$

  $x_3^* \wedge z_1^* + x_2^* \wedge z_2^*$

  $x_3^* \wedge z_2^* + x_2^* \wedge z_1^*$

  $-x_1^* \wedge z_1^* + x_4^* \wedge z_2^*$

- $H^3(L)$:

$x_2^* \wedge x_3^* \wedge z_1^*$

$x_1^* \wedge x_4^* \wedge z_1^*$

$x_2^* \wedge x_4^* \wedge z_1^* - x_1^* \wedge x_3^* \wedge z_1^*$

$x_3^* \wedge x_4^* \wedge z_1^* - x_1^* \wedge x_2^* \wedge z_1^*$

$x_1^* \wedge x_3^* \wedge z_1^* + x_1^* \wedge x_2^* \wedge z_2^*$

$x_1^* \wedge x_2^* \wedge z_1^* + x_1^* \wedge x_3^* \wedge z_2^*$

$x_2^* \wedge x_3^* \wedge z_2^*$

$x_1^* \wedge x_4^* \wedge z_2^*$

$x_2^* \wedge x_4^* \wedge z_2^* + x_1^* \wedge x_2^* \wedge z_1^*$

$x_3^* \wedge x_4^* \wedge z_2^* + x_1^* \wedge x_3^* \wedge z_1^*$

- $H^4(L)$:

$-x_2^* \wedge x_3^* \wedge x_4^* \wedge z_1^* + x_1^* \wedge x_2^* \wedge x_3^* \wedge z_2^*$

$x_1^* \wedge x_3^* \wedge x_4^* \wedge z_1^* + x_1^* \wedge x_2^* \wedge x_4^* \wedge z_2^*$

$x_1^* \wedge x_2^* \wedge x_4^* \wedge z_1^* + x_1^* \wedge x_3^* \wedge x_4^* \wedge z_2^*$

$x_2^* \wedge x_3^* \wedge x_4^* \wedge z_2^* - x_1^* \wedge x_2^* \wedge x_3^* \wedge z_1^*$

$x_2^* \wedge x_3^* \wedge z_1^* \wedge z_2^*$

$x_1^* \wedge x_4^* \wedge z_1^* \wedge z_2^*$

$x_2^* \wedge x_4^* \wedge z_1^* \wedge z_2^* - x_1^* \wedge x_3^* \wedge z_1^* \wedge z_2^*$

$x_3^* \wedge x_4^* \wedge z_1^* \wedge z_2^* - x_1^* \wedge x_2^* \wedge z_1^* \wedge z_2^*$

- $H^5(L)$:

$x_1^* \wedge x_2^* \wedge x_3^* \wedge z_1^* \wedge z_2^*$

$x_1^* \wedge x_2^* \wedge x_4^* \wedge z_1^* \wedge z_2^*$

$x_1^* \wedge x_3^* \wedge x_4^* \wedge z_1^* \wedge z_2^*$

$x_2^* \wedge x_3^* \wedge x_4^* \wedge z_1^* \wedge z_2^*$

- $H^6(L)$:

$x_1^* \wedge x_2^* \wedge x_3^* \wedge x_4^* \wedge z_1^* \wedge z_2^*$

3.3 The Laplacian operator

The goal of this section is to recall the definition of the Laplacian, a linear operator $\Delta : \Lambda L^* \to \Lambda L^*$ such that $\ker \Delta \cong H^*(L)$. This represents the cohomology as the so-called *harmonic forms* ([16]).

We begin by introducing an inner product on ΛV for an arbitrary inner product space V with $n := \dim V$. We fix an ordered orthonormal basis $\{e_i\}_{1 \le i \le n}$ of V.

Definition 3.3.1. The Hodge star map $\star : \Lambda V \to \Lambda V$ is defined on basis elements and extended linearly as follows. Consider a typical basis element $e = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ where $i_1 < i_2 < \cdots < i_p$. Let $e' = e_{j_1} \wedge \cdots \wedge e_{j_q}$ be the wedge of all basis elements not appearing in the list $i_1, i_2, \ldots, i_p$, with $q = n - p$. Now we can define:

$$\star e = (-1)^{\operatorname{sgn}(i_1, i_2, \ldots, i_p, j_1, \ldots, j_q)} e'$$

where $\operatorname{sgn}(a_1, a_2, \ldots, a_n)$ is the sign of the permutation mapping l to a_l.

Remark 3.3.2. Given $\alpha \in \Lambda^p V$, straightforward calculations yield that $\star\alpha \in \Lambda^{n-p} V$ and $\star\star\, \alpha = (-1)^{p(n-p)} \alpha$.

Definition 3.3.3. As in [2], we use the Hodge star to endow ΛV with the structure of an inner product space. If $x, y \in \Lambda^p V$, then we define $\langle x, y \rangle_\star := \star(x \wedge \star y)$, otherwise $\langle x, y \rangle_\star := 0$.

Remark 3.3.4. We can show that $\langle , \rangle_\star$ is an inner product as follows. It is clearly linear. Now, let $x := e_{i_1} \wedge \cdots \wedge e_{i_p}$ and $y := e_{j_1} \wedge \cdots \wedge e_{j_p}$ be monomials, with $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$. If for some k, $i_k \neq j_k$, then $e_{j_k} \wedge x \neq 0$, and $e_{j_k} \wedge \star y = 0$. Because y lies in the ideal generated by e_{j_k},

$$\langle x, y \rangle = \star(x \wedge \star y) = 0.$$

On the other hand, if for all k, $i_k = j_k$, then $x = y$, and by the properties of the Hodge star map,

$$\langle x, y \rangle_\star = \star(x \wedge \star y) = 1$$

Therefore the set of monomials forms an orthonormal basis for ΛL^*, and so $\langle , \rangle_\star$ is an inner product.

Now we return to Lie algebras. In the remainder of this section, we assume that L is a unimodular Lie algebra[1].

Lemma 3.3.5. *The restriction of d to $\Lambda^{\dim L - 1} L^*$ is zero.*

Proof. We choose dual bases $\{x_1, \dots, x_n\}$ and $\{x_1^*, \dots, x_n^*\}$ where $n = \dim L$. We write dx_i^* relative to this basis:

$$dx_i^* := \sum_{1 \le j < k \le n} \beta_{j,k}^i x_j^* \wedge x_k^*$$

For $j > k$, set $\beta_{j,k}^i := -\beta_{k,j}^i$. Set $\beta_{j,j}^i = 0$ for all j.

Note that for any linear operator $T : L^* \to L^*$, we have

$$\operatorname{tr} T = \sum_{i=1}^n i_{x_i} T(x_i^*)$$

Hence, we perform a calculation:

$$\begin{aligned} \operatorname{tr}(i_x d) &= \sum_{i=1}^n i_{x_i}(i_x dx_i^*) \\ &= \sum_{i=1}^n \sum_{1 \le j < k \le n} i_{x_i}(i_x(\beta_{j,k}^i x_j^* \wedge x_k^*)) \\ &= -\sum_{i=1}^n \sum_{1 \le j < k \le n} i_x(i_{x_i}(\beta_{j,k}^i x_j^* \wedge x_k^*)) \\ &= \sum_{i=1}^n \sum_{k=1}^n i_x(\beta_{i,k}^i x_k^*) \end{aligned}$$

Define $\gamma \in L^*$ by $\gamma(x) := \operatorname{tr} \operatorname{ad} x$. Since L is unimodular, $\gamma = 0$; on the other hand, the above calculation and lemma 3.1.3 shows

$$\gamma = \sum_{i=1}^n \sum_{k=1}^n \beta_{i,k}^i x_k^*$$

[1] In fact a slightly more detailed approach allows one to define the Laplacian for arbitrary Lie algebras L; the reader is referred to [14] or [2] for the details, as we do not need those results here.

Now, we pick an element $\alpha \in \Lambda^{n-1}L^*$, and show that $d\alpha = 0$. Without loss of generality, we can assume α is a monomial, and furthermore, we can assume $\alpha := x_1^* \wedge \cdots \wedge x_{n-1}^*$; the argument for the other monomials in the canonical basis of $\Lambda^{n-1}L^*$ is similar. We compute:

$$\begin{aligned} d\alpha &= \sum_{1 \le i \le n-1} (-1)^{i+1} dx_i^* \wedge x_1^* \wedge \cdots \wedge \hat{x_i^*} \wedge \cdots \wedge x_{n-1}^* \\ &= \sum_{1 \le i \le n-1} (-1)^{i+1} \beta_{i,n}^i x_i^* \wedge x_n^* \wedge x_1^* \wedge \cdots \wedge \hat{x_i^*} \wedge \cdots \wedge x_{n-1}^* \\ &= -\left(\sum_{1 \le i \le n-1} \beta_{i,n}^i x_n^* \right) \wedge x_1^* \wedge \cdots \wedge x_{n-1}^* \\ &= -\gamma \wedge \alpha \\ &= 0. \end{aligned}$$

□

Definition 3.3.6. The adjoint of the differential d with respect to the inner product $\langle , \rangle_\star$ is a linear map of degree -1 denoted $\partial : \Lambda L^* \to \Lambda L^*$.

Remark 3.3.7. If $\chi : V \to V^*$ is the isomorphism induced by the inner product on V, then:

$$\begin{aligned} \langle z, \partial(u \wedge v) \rangle_\star &= \langle dz, u \wedge v \rangle_\star \\ &= \langle dz, \chi(u) \wedge \chi(v) \rangle \\ &= \langle z, [\chi(u), \chi(v)] \rangle \end{aligned}$$

Thus $\partial(u \wedge v) = [\chi(u), \chi(v)]$, showing that ∂ is the Lie bracket on $\Lambda^2 V^*$.

Proposition 3.3.8. *On $\Lambda^p L^*$, $\partial = (-1)^{n(p+1)+1} \star d \star$, where $n = \dim L$.*

Proof. Let $\alpha \in \Lambda^{p-1}L^*$, $\beta \in \Lambda^p L^*$. Since the restriction of d to $\Lambda^{\dim L - 1}L^*$ is zero, we have that

$$\begin{aligned} 0 &= d(\alpha \wedge \star\beta) \\ &= d\alpha \wedge \star\beta + (-1)^{p-1} \alpha \wedge (d \star \beta) \end{aligned}$$

This is equivalent to:

$$d\alpha \wedge \star\beta = (-1)^p \alpha \wedge (d \star \beta)$$

Now we compute:

$$\begin{aligned}
\langle \alpha, (-1)^{n(p+1)+1} \star d \star \beta \rangle_\star &= (-1)^{n(p+1)+1} \star (\alpha \wedge \star \star d \star \beta) \\
&= (-1)^{n(p+1)+1+(p-1)(n-p+1)} \star (\alpha \wedge d \star \beta) \\
&= (-1)^p \star (\alpha \wedge d \star \beta) \\
&= \star(d\alpha \wedge \star\beta) \\
&= \langle d\alpha, \beta \rangle_\star
\end{aligned}$$

This gives us our result, since we have

$$\langle \alpha, \partial\beta \rangle_\star = \langle \alpha, (-1)^{n(p+1)+1} \star d \star \beta \rangle_\star.$$

□

Definition 3.3.9. The Laplacian is a linear map of degree zero, $\Delta : \Lambda L^* \to \Lambda L^*$, defined as $\Delta = \partial d + d\partial$.

Theorem 3.3.10 (Hodge decomposition theorem for Lie algebras, [2])**.** *The Laplacian satisfies the following properties:*

1. *The Laplacian is self-adjoint.*
2. *There is an orthogonal decomposition* $\ker \Delta \oplus \operatorname{im} \Delta = \Lambda L^*$.
3. $\ker \Delta = \ker d \cap \ker \partial$.
4. *There is an orthogonal decomposition* $\ker d = \ker \Delta \oplus \operatorname{im} d$.
5. *There exists a vector space isomorphism* $H^*(L) \cong \ker \Delta$.

Proof. 1. If we denote the adjoint of a linear map T by $\overline{T}$, then adjoint of the Laplacian under the inner product $\langle , \rangle_\star$ is given by:

$$\overline{\Delta} = \overline{\partial d} + \overline{d\partial} = \partial d + d\partial,$$

so Δ is self-adjoint.

2. It is a standard result from linear algebra that if T is a linear operator, then $\ker T = (\operatorname{im} \overline{T})^{\perp}$. In our case, this tells us that $\ker \Delta = (\operatorname{im} \Delta)^{\perp}$, therefore $\Lambda L^* = \ker \Delta \oplus \operatorname{im} \Delta$, and furthermore this decomposition is orthogonal.

3. Note that $\langle \partial x, dy \rangle = \langle x, d^2 y \rangle = 0$, so $\operatorname{im} \partial \perp \operatorname{im} d$. If $x \in \ker \Delta$, then $\partial dx + d\partial x = 0$ and thus $\partial dx = 0$ and $d\partial x = 0$. Now, $\ker \partial \perp \operatorname{im} d$ and $\ker d \perp \operatorname{im} \partial$, hence $\partial dx = 0$ if and only if $dx = 0$, and $d\partial x = 0$ if and only if $\partial x = 0$. Therefore $x \in \ker d \cap \ker \partial$, and so $\ker \Delta \subseteq \ker d \cap \ker \partial$. The other inclusion is obvious, since if $x \in \ker d \cap \ker \partial$, $d\partial x = 0$ and $\partial dx = 0$, so $\Delta x = 0$.

4. Since $x \in \ker \partial$ if and only if $x \perp \operatorname{im} d$, and $\ker \Delta = \ker d \cap \ker \partial$, it follows that $\ker d = \ker \Delta \oplus \operatorname{im} d$.

5. Since $\ker d = \ker \Delta \oplus \operatorname{im} d$, $\ker d / \operatorname{im} d \cong \ker \Delta$.

□

3.4 Primary operations and ΛZ-modules

In the following, let Z be the centre of L. Recall that if R and S are algebras over the same field, any algebra homomorphism $\rho : R \to \operatorname{End}(S)$ defines an R-module structure on S, where $\operatorname{End}(S)$ is the algebra of linear maps on S under composition. We will thus equip $H^*(L)$ with a ΛZ-module structure for a Lie algebra L by defining a map $i_\zeta : H^*(L) \to H^*(L)$ for each $\zeta \in \Lambda Z$.

The following result is elementary but we include its proof for completeness.

Proposition 3.4.1. *For $z \in L$, the derivation i_z induces a well-defined map $H^*(i_z)$ if and only if $z \in Z$.*

Proof. ($\Rightarrow$) If $H^*(i_z)$ is well-defined, then in particular, for $\omega \in L^*$, $i_z d\omega \in \operatorname{im} d$.

For $x \in L$, we have

$$\begin{aligned} \langle i_z d\omega, x \rangle &= \langle d\omega, z \wedge x \rangle \\ &= \langle \omega, [z, x] \rangle. \end{aligned}$$

On the other hand, since $i_z d\omega \in \operatorname{im} d$, we must have that $i_z d\omega = 0$, since there are no non-zero boundaries of degree 1. Since ω was arbitrary, we see that $[z, x] = 0$ for all $x \in L$, or in other words, $z \in Z$.

($\Leftarrow$) Suppose that $z \in Z$. It follows that $(\operatorname{ad} z)^* = 0$, and from lemma 3.1.3, this is equivalent to $i_z d = 0$. In degree 1, $i_z d = [d, i_z]$. Since $[d, i_z]$ is a derivation which equals $i_z d$ on degree 1 elements, proposition 2.3.6 implies that $[d, i_z] = 0$. ☐

Definition 3.4.2. A *primary operation* on $H^*(L)$, denoted $H^*(i_z)$, is the map induced on cohomology by the derivation i_z for some $z \in Z$. This is also a derivation of the algebra $H^*(L)$. The primary operation is *non-trivial* if $H^*(i_z)$ is a non-zero map on cohomology.

Definition 3.4.3 ([4] page 1)**.** The linear map $i : Z \to \operatorname{End} H^*(L)$ which maps $z \in Z$ to $H^*(i_z)$ can be inductively extended to a linear map

$$i : \Lambda Z \to \operatorname{End} H^*(L)$$

by $i_{a\wedge b}(x) = i_a(i_b(x))$ where $|a| = 1$. Note that i is then a homomorphism of algebras, since for $|x|, |y| = 1$, $i_{x\wedge y} = i_x i_y$, and $i_x i_y = -i_y i_x$. This map i then gives $H^*(L)$ the structure of a ΛZ-module, defined by

$$z \cdot \omega := i_z(\omega)$$

Definition 3.4.4. If ΛZ contains an element ζ with $\deg \zeta \geq 1$ such that $H^*(i_\zeta) \neq 0$, then we say $H^*(L)$ possesses *non-trivial* ΛZ-module structure.

We now prove the theorem mentioned in the introduction.

Theorem 3.4.5. *[4] Let L be a Lie algebra, let Z be the centre of L, and let $0 \neq \tau \in \Lambda^{\dim Z} Z$. The following are equivalent:*

1. *There exists an element $\omega \in H^*(L)$ such that $i_\tau(\omega) \neq 0$.*

2. *The algebra homomorphism $i : \Lambda Z \to \operatorname{End}(H^*(L))$ is injective.*

3. *$H^*(L)$ contains a free ΛZ-submodule.*

Moreover if these conditions hold, then L satisfies the nil-TRC.

Proof. (1) $\Rightarrow$ (2) For any non-zero $\zeta \in \Lambda Z$, we have $\langle \zeta, \zeta \rangle_\star \neq 0$, hence $\zeta \wedge \star \zeta$ is a non-zero scalar multiple of τ and $H^*(i_\zeta \circ i_{\star\zeta})(\omega) \neq 0$, and in particular $H^*(i_\zeta)(\omega) \neq 0$, thus i is injective.

(2) $\Rightarrow$ (3) Since i is injective, there exists $\omega \in H^*(L)$ with $H^*(i_\tau)(\omega) \neq 0$. A similar argument to the above shows that for arbitrary $\zeta \in \Lambda Z$, $H^*(i_\zeta)(\omega) \neq 0$, thus ω generates a ΛZ-submodule of $H^*(L)$, necessarily free since it is isomorphic to ΛZ.

(3) $\Rightarrow$ (1) Let $M \subset H^*(L)$ be a free ΛZ-module. Pick any element $\omega \in M$ such that $i_\tau(\omega) \neq 0$.

To see that L satisfies the nil-TRC, note that $H^*(L)$ contains a free ΛZ-submodule, so

$$2^{\dim Z} = \dim \Lambda Z \leq \dim H^*(L).$$

$\square$

Not every nilpotent Lie algebra enjoys a cohomology space containing free ΛZ-submodules – see the next section for an example. Furthermore, if the nilpotency condition is relaxed to solvability, the ΛZ-module structure may in fact be trivial. This will clearly occur if L has a trivial centre; but even if $Z \neq 0$, there are a solvable Lie algebras whose cohomology has a trivial ΛZ-module structure. One such example is the 4-dimensional Lie algebra

$$L := \operatorname{span}\{z, a, b, x | [a, b] = z, [a, x] = a, [x, b] = b\}$$

an easy calculation shows $z \in Z$ but $H^*(i_z) = 0$. This simplifies a 7-dimensional example given in [4].

The situation with nilpotent Lie algebras is more interesting. In every example which has been calculated so far, the cohomology enjoys non-trivial ΛZ-module structure. This led Cairns and Jessup to pose the following conjecture in [4]:

Conjecture 3.4.6. *For any nilpotent Lie algebra L, $H^*(L)$ has a non-trivial ΛZ-module structure.*

We see that this conjecture is trivially true when L is an abelian Lie algebra:

Example 3.4.7. Let L be an abelian Lie algebra. Then the differential d on ΛL^* is zero, and so $H^*(L) = \Lambda L^*$. Now if $z \in L$, then $i_z(z) = 1$. Since both z and 1 are cycles which are not boundaries, $H^*(i_z) \neq 0$ for all $z \in Z$. Here, $H^*(L)$ is a free ΛZ-module generated by a single element.

An abelian Lie algebra can be thought of as a "one-step" nilpotent Lie algebra. In section 4.5, we prove that the above conjecture holds for two-step nilpotent algebras, a new result.

Chapter 4

Two-step nilpotent Lie algebras

In sections 4.2, 4.3 and 4.4, we study three specific classes of nilpotent Lie algebras. In section 4.5 we show that the cohomology of any two-step Lie algebra has non-trivial ΛZ-submodules, a previously unknown result.

4.1 Two-step Koszul complexes

In this section, we fix some notation which we will use for the remainder of this chapter.

Consider a two-step nilpotent Lie algebra L with centre Z. Recall that this means $[L, L] \subseteq Z$, and we can write $L = U \oplus Z$, with $U = \operatorname{span}\{x_i | 1 \leq i \leq \dim U\}$ and $Z = \operatorname{span}\{z_j | 1 \leq j \leq \dim Z\}$. We will choose corresponding dual bases of U^* and Z^*, and construct the exterior algebra ΛL^*.

We observe that the differential d is zero on U^*, and maps Z^* into $\Lambda^2 U^*$. To prove the first assertion, pick $x \in U^*$, $a, b \in L$, and calculate the pairing

$$\langle dx, a \wedge b \rangle = \langle x, [a, b] \rangle.$$

Since $[a, b] \in Z$ and every element of U^* annihilates Z, $\langle x, [a, b] \rangle = 0$.

To see that for any $x \in Z^*$, $dx \in \Lambda^2 U^*$, observe that if $z \in Z$ and $y \in L$, then $[z, y] = 0$, therefore for any $x \in L^*$, $\langle dx, z \wedge y \rangle = 0$. Thus the boundary of an element of L^* annihilates $U \otimes Z$ and $\Lambda^2 Z$, and thus must lie in $\Lambda^2 U^*$.

We end this section with three remarks.

Remark 4.1.1. There is a notion of a *bigraded vector space*, completely analogous to a graded vector space.

For a two-step Lie algebra $L = U \oplus Z$, ΛL^* is bigraded; we have

$$\Lambda L^* \cong \Lambda U^* \otimes \Lambda Z^*$$

and

$$(\Lambda L^*)^{p,q} \cong \Lambda^{p-q} U^* \otimes \Lambda^q Z^*.$$

An element $\omega \in \Lambda^{p-q} U^* \otimes \Lambda^q Z^*$ is said to have bidegree (p, q). We note that the differential d is a derivation of bidegree $(1, -1)$, ie:

$$d : \Lambda^{p-q} U^* \otimes \Lambda^q Z^* \to \Lambda^{p-q+2} U^* \otimes \Lambda^{q-1} Z^*.$$

This follows from the Leibnitz rule and an induction argument, since $d|_{Z^*} : Z^* \to \Lambda^2 U^*$.

It follows that $\ker d$ and $\operatorname{im} d$ are bigraded vector spaces, and so $H^*(L)$ is also bigraded.

Remark 4.1.2. The previous remark can be used to give an alternative proof of lemma 3.3.5 in the two-step case. Indeed, if $m := \dim U$ and $n := \dim Z$, the differential maps

$$d : \Lambda^{m-2} U^* \otimes \Lambda^{n+1} Z^* \to \Lambda^m U^* \otimes \Lambda^n Z^*$$

but $\Lambda^{n+1} Z^* = 0$. Hence no non-zero element of $\Lambda^m U^* \otimes \Lambda^n Z^* \cong \Lambda^{\dim L} L^*$ lies in the image of d.

Remark 4.1.3. The Hodge star on ΛL^*, denoted $\star$, is defined relative to the dual basis $\{x_1^*, \ldots, x_m^*, z_1^* \ldots z_n^*\}$. This turns ΛL^* into an inner product space and allows us to define the Laplacian Δ. The Hodge star on ΛU^* relative to U's ordered basis will be denoted $\underline{\star}$. Note that on ΛU^*, we have:

$$\star x = \pm z_1^* \wedge \cdots \wedge z_n^* \wedge \underline{\star} x$$

This fact is used in the proof of theorem 4.3.4.

Remark 4.1.4. Suppose we wish to find an element $\omega \in \Lambda U^* \subset \Lambda L^*$ which is a cycle but not a boundary. One approach is to look for ω satisfying $\Delta\omega = 0$, since $\ker \Delta \cong H^*(L)$. By theorem 3.3.10, $\omega \in \ker \Delta$ if and only if:

$$\begin{aligned} d\omega &= 0 \\ d(\star\omega) &= 0. \end{aligned}$$

The first condition gives us nothing new, since $d(\Lambda U^*) = 0$. By remark 4.1.3, the second condition becomes:

$$d(z_1 \wedge \cdots \wedge z_n \wedge \underline{\star}\omega) = 0$$

The Leibnitz rule now shows that $\omega \in \ker \Delta$ if and only if $dz_i \wedge \underline{\star}\omega = 0$ for all i.[1] This fact is used in the proof of theorem 4.4.1.

4.2 Free two-step nilpotent Lie algebras

The cohomology of free two-steps has been extensively studied and much is known; for example see [16]. In this section we study the ΛZ-module structure of the cohomology of free two-steps. We show that no free submodules exist, and that the module structure is non-trivial.

Definition 4.2.1. As a vector space, F_n, the nth *free two-step nilpotent Lie algebra*, is defined as

$$F_n := U \oplus Z$$

where $U := \operatorname{span}\{e_i | 1 \leq i \leq n\}$ and $Z := \operatorname{span}\{f_{ij} | 1 \leq i < j \leq n\}$. The Lie brackets are given by $[e_i, e_j] = f_{ij}$ for $i < j$; all other brackets follow from anti-symmetry. Note that $\dim Z = \binom{n}{2}$.

[1]While the above calculation ensures that a *sufficient* condition for an element $x \in \Lambda U^*$ to not be a boundary is to have $\partial x = 0$, this is not a *necessary* condition; clearly, $\partial x = 0$ if and only if $x \in (\operatorname{im} d)^\perp$, but it may happen that $x \notin \operatorname{im} d$ and $x \notin (\operatorname{im} d)^\perp$.

If we choose a dual basis of F_n, say

$$\{e_i^*|1 \le i \le n\} \cup \{f_{ij}^*|1 \le i < j \le n\}$$

it is easy to see that $d(f_{ij}^*) = e_i^* \wedge e_j^*$ for $i < j$.

Proposition 4.2.2. *For $n > 1$, the cohomology space $H^*(F_n)$ does not contain any non-zero free ΛZ-submodules.*

Proof. By 3.4.5, it follows that $H^*(F_n)$ contains a free ΛZ-submodule if and only if there exists $\omega \in H^*(F_n)$ such that $i_\tau(\omega) \neq 0$, where τ is a non-zero element of $\Lambda^{\dim Z} Z$.

Let w be a representative of the class ω in ΛF_n^*. We must necessarily have $w = u \wedge \tau^*$, where $u \in \Lambda U^*$ and τ^* is a non-zero element of $\Lambda^{\dim Z} Z^*$. Since $H^*(i_\tau) \neq 0$, we have $u \notin \operatorname{im} d$. Since $\Lambda^{\ge 2} F_n^* \subset \operatorname{im} d$, $|u| = 1$.

Furthermore, since

$$\tau^* = f_{12}^* \wedge f_{13}^* \wedge \cdots \wedge f_{(n-1)n}^*,$$

it follows by the Leibnitz rule that for all $1 \le i < j \le n$,

$$d(u \wedge f_{ij}^*) = \pm u \wedge e_i^* \wedge e_j^*.$$

In particular,

$$\begin{aligned} u \wedge e_1^* \wedge e_2^* &= 0 \\ u \wedge e_1^* \wedge e_3^* &= 0 \\ u \wedge e_2^* \wedge e_3^* &= 0. \end{aligned}$$

However, this is equivalent to

$$\begin{aligned} u &\in \operatorname{span}\{e_1^*, e_2^*\} \\ u &\in \operatorname{span}\{e_1^*, e_3^*\} \\ u &\in \operatorname{span}\{e_2^*, e_3^*\}. \end{aligned}$$

Therefore, we may conclude that

$$u \in \text{span}\{e_1^*, e_2^*\} \cap \text{span}\{e_1^*, e_3^*\} \cap \text{span}\{e_2^*, e_3^*\} = 0.$$

This contradicts our assumption that $i_\tau(\omega) \neq 0$, and so no free ΛZ-submodules exist. □

Remark 4.2.3. In the case $n = 1$, F_1 is simply the Heisenberg Lie algebra $\mathfrak{h}_1$ as defined in example 2.1.11. Corollary 4.3.5 shows that $H^*(\mathfrak{h}_1)$ contains a direct sum of two free ΛZ-modules.

In section 4.5 we will see that all two-steps enjoy a non-trivial ΛZ-module structure. In the case of F_n, we can do better than the generic result by exploiting the structure of F_n.

First, we recall *Engel's theorem.*

Theorem 4.2.4. *Let L be a nilpotent Lie algebra. Then there exists a basis $\{x_i\}_{1\leq i\leq n}$ such that for i, j satisfying $1 \leq i < j \leq n$, $[x_i, x_j] \in \text{span}\{x_k | j + 1 \leq k \leq n\}$.*

An equivalent formulation of the above is the following:

Theorem 4.2.5. *Let $(\Lambda L^*, d)$ be the Koszul complex of a nilpotent Lie algebra. Then there exists a basis $\{x_i^*\}_{1\leq i\leq n}$ of L^* such that for i satisfying $1 \leq i \leq n$, $dx_i^* \in \Lambda^2\text{span}\{x_k^* | 1 \leq k < i\}$.*

Now, we present a result showing that a certain class of nilpotent Lie algebras satisfies conjecture 3.4.6. This class contains representatives of arbitrarily high nilpotency degree.

Theorem 4.2.6. *Let L be a nilpotent Lie algebra, and let $(\Lambda L^*, d)$ be the corresponding Koszul complex. Suppose that there exists $z^* \in Z^*$ such that $dz^* = a^* \wedge b^*$, with a^* and b^* linearly independent, and let $z \in Z$ satisfy $\langle z^*, z\rangle = 1$. Then $H^*(i_z) \neq 0$.*

Proof. If $da^* = 0$, then $i_z(z^* \wedge a^*) = a^* \notin \operatorname{im} d$, and hence $H^*(i_z) \neq 0$. A similar argument can be made if $db^* = 0$.

So assume that neither a^* nor b^* are elements of $\ker d$. We show this leads to a contradiction.

We differentiate z^* twice:

$$d^2 z^* = da^* \wedge b^* - a^* \wedge db^*$$

However, $d^2 = 0$, so we have

$$da^* \wedge b^* = a^* \wedge db^*.$$

We may write

$$\begin{aligned} da^* &= v_2^* \wedge a^* + v_1^* \wedge b^* \\ db^* &= v_3^* \wedge a^* + v_4^* \wedge b^* \end{aligned}$$

where $v_i \in U^*$.

Let $\{x_i^*\}_{1 \leq i \leq n}$ be an Engel basis for L^*. Without loss of generality, we may assume $x_i^* = a^*$ and $x_j^* = b^*$, with $i < j$. But now, we have a contradiction, since for i satisfying $1 \leq i \leq n$, $dx_i^* \in \Lambda^2 \text{span}\{x_k^* | 1 \leq k < i\}$.

Therefore one of da^* or db^* is zero, and so $H^*(i_z) \neq 0$. □

Now we apply the above result to F_n.

Theorem 4.2.7. *Let F_n be the free two-step nilpotent Lie algebra with n generators and centre Z. Then for all non-zero $z \in Z$, $H^*(i_z) \neq 0$.*

Proof. For any $1 \leq i < j \leq n$, $df_{ij}^* = e_i^* \wedge e_j^*$. Therefore, by theorem 4.2.5, $H^*(i_{f_{ij}}) \neq 0$. For a non-zero $z \in Z$, we write

$$z = \sum_{1 \leq i < j \leq n} c_{ij} f_{ij}.$$

Let i, j be the first index such that $c_{ij} \neq 0$; now,

$$\begin{aligned} i_z(f_{ij}^* \wedge e_i^*) &= i_{f_{ij}}(f_{ij}^* \wedge e_i^*) + i_{z-f_{ij}}(f_{ij}^* \wedge e_i^*) \\ &= e_i^* + 0 \\ &= e_i^* \notin \operatorname{im} d \end{aligned}$$

□

Note that while theorem 4.5.1 only shows that $H^*(i_z) \neq 0$ for *some* $z \in Z$, here we are able to show that $H^*(i_z) \neq 0$ for *all* non-zero $z \in Z$.

4.3 Free ΛZ-submodules, example I

In this section, we study a class of Lie algebras whose cohomology enjoys some interesting properties and in particular, contains free ΛZ-submodules.

For the remainder of this section, suppose $L = U \oplus Z$ is a two-step Lie algebra, with U even-dimensional, $m := \frac{\dim U}{2}$ and $n := \dim Z$.

The results in sections 2.4 and 2.5 will be crucial here. In particular, recall theorem 2.5.1, which constructs a vector space isomorphism:

$$\eta : \mathfrak{so}(U) \to \Lambda^2 U^*$$

We begin with some equivalent characterizations of a certain class of two-step Lie algebras.

Theorem 4.3.1. *Let $\{z_i | 1 \leq i \leq n\}$ and $\{z_i^* | 1 \leq i \leq n\}$ be dual bases of Z and Z^*. The following are equivalent:*

1. *All elements in the vector space $\{\eta^{-1}(dz^*) | z^* \in Z^*\} \subset \mathfrak{so}(U)$ commute under the Lie bracket on $\mathfrak{so}(U)$.*

2. *There exists a basis of* U^*, $\{u_i^*, v_i^*\}_{1\leq i\leq m}$, *and a set of linear functionals* $\{c_i\}_{1\leq i\leq n} \subset Z^{**}$, *such that every* $z^* \in Z^*$ *has a boundary of the following form with respect to this basis:*

$$dz^* = \sum_{1\leq i\leq m} c_i(z) u_i^* \wedge v_i^*$$

3. *There exists a basis of* U^*, $\{u_i^*, v_i^*\}_{1\leq i\leq m}$ *and a set of vectors in* Z, $\{\zeta_i\}_{1\leq i\leq m}$ *(not necessarily linearly independent or spanning* Z*) such that the differential takes the following form:*

$$d = \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge i_{\zeta_i}$$

Proof. (1) $\Leftrightarrow$ (2) This is corollary 2.5.3 together with an induction on n.

(2) $\Rightarrow$ (3) Assume that we have a basis $\{u_i^*, v_i^*\}_{1\leq i\leq m}$ such that

$$dz^* = \sum_{1\leq i\leq m} c_i(z) u_i^* \wedge v_i^*$$

for every $z \in Z^*$. This implies that for any $\omega \in \Lambda L^*$,

$$\begin{aligned} d\omega &= \sum_{1\leq j\leq n} dz_j^* \wedge i_{z_j}(\omega) \\ &= \sum_{1\leq j\leq n} \sum_{1\leq i\leq m} c_i(z_j^*) u_i^* \wedge v_i^* \wedge i_{z_j}(\omega) \\ &= \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge \sum_{1\leq j\leq n} c_i(z_j^*) i_{z_j}(\omega) \\ &= \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge i_{\zeta_i}(\omega) \end{aligned}$$

where

$$\zeta_i = \sum_{1\leq j\leq n} c_i(z_j^*) z_j$$

(3) $\Rightarrow$ (2) Note that the hypothesis gives

$$dz^* = \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge i_{\zeta_i}(z).$$

Define $c_i(z) := i_{\zeta_i}(z)$. The result follows immediately.

$\square$

Remark 4.3.2. There is no loss of generality in assuming that $\dim U$ is even; it is not difficult to see that if $\dim U$ were odd, condition (3) in the above theorem yields a basis element $x \in U$ with $i_x d = 0$; but lemma 3.1.3 now forces $x \in Z$, a contradiction.

The following lemma establishes the behaviour of this class of Lie algebras under two-dimensional extensions. We shall use it to prove a key structure result.

Lemma 4.3.3. *Suppose the following hold:*

1. $\overline{L} = \overline{U} \oplus Z$ *satisfies the equivalent conditions of theorem 4.3.1.*

2. $L = \overline{L} \oplus \operatorname{span}\{u, v\}$, $U = \overline{U} \oplus \operatorname{span}\{u, v\}$.

3. *The differential d on ΛL^* is zero on u^* and v^*, and there is $\zeta \in Z$ such that on other elements d is given by*

$$d = \overline{d} + u^* \wedge v^* \wedge i_\zeta$$

where $\overline{d}$ is the differential on $\Lambda\overline{L}$.

4. *There exists a $\overline{d}$-cycle $\omega \in \Lambda^p \overline{U}^* \otimes \Lambda^q Z^*$, which is non-zero in $H^*(\overline{L})$, for some p and q.*

Then there exists a d-cycle $\omega' \in \Lambda^{p+2} U^* \otimes \Lambda^{q'} Z^*$, with $q \leq q' \leq q+1$, which is non-zero in $H^*(L)$.

Proof. Note that $d(u^* \wedge v^* \wedge \omega) = 0$. If $u \wedge v \wedge \omega \notin \operatorname{im} d$, we are done, so assume $u^* \wedge v^* \wedge \omega = d\alpha$ for some $\alpha \in \Lambda^p U^* \otimes \Lambda^{q+1} Z^*$. Again we observe that $d(u^* \wedge v^* \wedge \alpha) = 0$. If $u^* \wedge v^* \wedge \alpha \notin \operatorname{im} d$, we are done, so assume $u^* \wedge v^* \wedge \alpha = d\beta$ for some $\beta \in \Lambda^p U^* \otimes \Lambda^{q+2} Z^*$. We show that this leads to a contradiction, finishing the proof.

Write α as follows:

$$\alpha = \alpha_1 + u^* \wedge \alpha_2 + v^* \wedge \alpha_3 + u^* \wedge v^* \wedge \alpha_4$$

where

$$\begin{aligned}\alpha_1 &\in \Lambda^p \overline{U}^* \otimes \Lambda^{q+1} Z^* \\ \alpha_2, \alpha_3 &\in \Lambda^{p-1} \overline{U}^* \otimes \Lambda^{q+1} Z^* \\ \alpha_4 &\in \Lambda^{p-2} \overline{U}^* \otimes \Lambda^{q+1} Z^*\end{aligned}$$

We compute the boundary of α:

$$\begin{aligned}d\alpha &= \overline{d}\alpha_1 + u^* \wedge v^* \wedge i_\zeta \alpha_1 - u^* \wedge \overline{d}\alpha_2 - v^* \wedge \overline{d}\alpha_3 + u^* \wedge v^* \wedge \overline{d}\alpha_4 \\ &= u^* \wedge v^* \wedge \omega\end{aligned}$$

So,

$$i_{u\wedge v} d\alpha = i_\zeta \alpha_1 + \overline{d}\alpha_4.$$

On the other hand, $d\alpha = u^* \wedge v^* \wedge \omega$ and $\omega \in \Lambda \overline{L}^*$, which gives us

$$i_{u\wedge v} d\alpha = \omega$$

hence

$$\omega = i_\zeta \alpha_1 + \overline{d}\alpha_4. \tag{4.3.1}$$

Next we decompose β just as we did with α above:

$$\beta = \beta_1 + u^* \wedge \beta_2 + v^* \wedge \beta_3 + u^* \wedge v^* \wedge \beta_4$$

Since $d\beta = u^* \wedge v^* \wedge \alpha$ and $i_{u\wedge v} d\beta = \alpha_1$, a calculation identical to 4.3.1 yields

$$\alpha_1 = i_\zeta \beta_1 + \overline{d}\beta_4.$$

Substituting the equation into 4.3.1 and using the fact that $i_\zeta^2 = 0$ now exhibits ω as a $\overline{d}$-boundary:

$$\begin{aligned}\omega &= i_\zeta \alpha_1 + \overline{d}\alpha_4 \\ &= i_\zeta(i_\zeta \beta_1 + \overline{d}\beta_4) + \overline{d}\alpha_4 \\ &= \overline{d}(\alpha_4 - i_\zeta \beta_4)\end{aligned}$$

Since we assumed $\omega \notin \operatorname{im} \overline{d}$, this contradicts the assumption that $u^* \wedge v^* \wedge \alpha \in \operatorname{im} d$, and so $u^* \wedge v^* \wedge \alpha$ is our non-trivial d-cycle. □

We now come to the main theorem of this section.

Theorem 4.3.4. *Assume L satisfies the equivalent conditions of theorem 4.3.1. Then the following hold:*

1. *$H^*(L)$ contains a direct sum of 2^n free ΛZ-submodules.*
2. *For every p with $0 \leq p \leq 2m$, there exists a cycle which is not a boundary in $(\Lambda^p U^* \otimes \Lambda Z^*, d)$.*
3. *$\dim H^*(L) \geq 2^m \cdot 2^n + 2m$.*

Proof. Recall that we defined $m := \frac{\dim U}{2}$ (remark 4.3.2 ensures this is an integer), and $n := \dim Z$.

1. For this part of the proof, we use the isomorphism $H^*(L) \cong \ker \Delta$ from section 3.3. Note that every i with $0 \leq i < 2^m$ has a binary expansion of not more than m digits. If the jth digit is 0, let $x_j = u_j^*$; otherwise, if the jth digit is 1, let $x_j = v_j^*$. Now define

$$\kappa_i = x_1 \wedge \cdots \wedge x_m$$

   Observe that both κ_i and $\star\kappa_i$ annihilate $u_j^* \wedge v_j^*$ for all j, therefore κ_i and $\star\kappa_i$ annihilate dz^* for all $z^* \in Z^*$. By the Leibnitz rule, it follows that for any $\zeta \in \Lambda Z^*$, $d(\kappa_i \wedge \zeta) = 0$, and from remark 4.1.3, we get $\partial(\kappa_i \wedge \zeta) = \pm\star d(\star(\kappa_i \wedge \zeta)) = 0$. By theorem 3.3.10, $\ker \Delta = \ker d \cap \ker \partial$, hence $\kappa_i \wedge \zeta$ is a harmonic form in $\ker \Delta$. This way we see that for every $0 \leq i < 2^m$, we get a free submodule K_i, generated by the harmonic elements $\{\kappa_i \wedge \zeta | \zeta = z_{i_1}^* \wedge \cdots \wedge z_{i_k}^*\}$, where ζ ranges over all monomials in the canonical basis of ΛZ^*.

   Since the κ_i are distinct monomials, the generators of all free submodules are linearly independent elements of $\ker \Delta \subset \Lambda L^*$. Because $\ker \Delta \simeq H^*(L)$, the generators are also linearly independent as elements of $H^*(L)$, and the sum of the free submodules is direct.

2. Write the differential as in part (3) of theorem 4.3.1:

$$d = \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge i_{\zeta_i}$$

Let $U_0^* := 0$, $U_k^* := \mathrm{span}\{u_1^*, v_1^*, \ldots, u_k^*, v_k^*\}$, and define a sequence of differentials:

$$\begin{aligned} d_0 &:= 0 \\ d_1 &:= u_1^* \wedge v_1^* \wedge i_{\zeta_1} \\ &\vdots \\ d_m &:= \sum_{1\leq i\leq m} u_i^* \wedge v_i^* \wedge i_{\zeta_i} \end{aligned}$$

Note that $d_m = d$.

Suppose ω is a non-trivial cycle in $(\Lambda^j U_k^* \otimes \Lambda^q Z^*, d_k)$. Now, we use lemma 4.3.3 with:

$$\begin{aligned} \overline{d} &:= d_k \\ u &= u_k \\ v &= v_k \end{aligned}$$

The lemma gives us a new non-trivial cycle in $(\Lambda^{j+2}U_k^* \otimes \Lambda Z^*, d_k)$. For any k, we can repeat this $m-k$ times, and get a non-trivial cycle in $(\Lambda^{j+2(m-k)}U^* \otimes \Lambda Z^*, d)$.

Now let j be the residue of p modulo 2, and let $k := m - \frac{p-j}{2}$. From remark 3.2.2, we have that $(\Lambda^j U_k^* \otimes \Lambda Z^*, d_k)$ contains a non-trivial cycle. The above observation yields a non-trivial cycle in U-degree $j + 2(m-k) = p$.

3. Part (1) gives us a collection of cycles $\{\kappa_i\}_{0\leq i<2^m}$ in ΛU^* and a monomial basis of ΛZ^*, $\{\zeta_j\}_{0\leq j<n}$ such that for every i, j, $\kappa_i \wedge \zeta_j$ is a non-trivial cycle. Part (2) gives us a collection of cycles $\{\omega_p\}_{0\leq p\leq 2m}$ with $\omega_p \in \Lambda^p U^* \otimes \Lambda Z^*$. Now for $p = m$, it may happen that $\omega_p = \kappa_i$ for some i; for all $p \neq m$, ω_p and κ_i's are

linearly independent for degree reasons. So we see that

$$\dim H^*(L) \geq 2^m \cdot 2^n + 2m.$$

In particular, L satisfies the Toral rank conjecture, and the dimension bound given in the TRC is a considerable underestimate in this case.

□

The above theorem allows us to prove a weaker form of a result from [14].

Corollary 4.3.5. *The cohomology of the Heisenberg Lie algebra $\mathfrak{h}_m$ contains a direct sum of two free ΛZ-submodules, and therefore $\mathfrak{h}_m$ satisfies the TRC.*

Proof. If $\dim Z = 1$, the conditions in theorem 4.3.1 are clearly satisfied, and theorem 4.3.4 applies. □

We note that our dimension estimate is slightly stronger than the one given for *all* two-step nilpotent Lie algebras in [17], but it still underestimates the total dimension, as the following example shows.

Example 4.3.6. Take the two-step nilpotent Lie algebra $L = U \oplus Z$ with $U = \text{span}\{u_i, v_i | 1 \leq i \leq 5\}$, $Z = \text{span}\{z_j | 1 \leq j \leq 3\}$, with non-zero brackets:

$$\begin{aligned}
[u_1, v_1] &= z_1 \\
[u_2, v_2] &= z_1 \\
[u_3, v_3] &= z_1 + z_2 \\
[u_4, v_4] &= z_2 + z_3 \\
[u_5, v_5] &= z_3
\end{aligned}$$

Choosing the dual basis in the usual way, we get the following values for the differential:

$$\begin{aligned}
dz_1^* &= u_1^* \wedge v_1^* + u_2^* \wedge v_2^* + u_3^* \wedge v_3^* \\
dz_2^* &= u_3^* \wedge v_3^* + u_4^* \wedge v_4^* \\
dz_3^* &= u_4^* \wedge v_4^* + u_5^* \wedge v_5^*
\end{aligned}$$

Table 4.1: Bigraded Betti numbers for $H^*(L)$, where p is the U-degree and q is the Z-degree

		q			
		0	1	2	3
p	0	1	0	0	0
	1	10	0	0	0
	2	42	14	0	0
	3	90	118	8	0
	4	92	286	105	0
	5	40	294	294	40
	6	0	105	286	92
	7	0	8	118	90
	8	0	0	14	42
	9	0	0	0	10
	10	0	0	0	1

We see that the dz_i commute, so L satisfies the conditions of theorem 4.3.1.

Computer calculations were used to compute the bigraded Betti numbers shown in table 4.3.6. Here the total dimension of $H^*(L)$ is 2688. The Toral Rank Conjecture only suggests that $H^*(L)$ should have dimension at least $2^3 = 8$. Part (3) in theorem 4.3.4 gives

$$\dim H^*(L) \geq 2^5 \cdot 2^3 + 10 - 1 = 265$$

which is still a considerable underestimate.

4.4 Free ΛZ-submodules, example II

For the remainder of this section, suppose $L = U \oplus Z$ is a two-step Lie algebra, with $U := \text{span}\{x_i | 1 \leq i \leq 2m\}$ for some m, and $Z := \text{span}\{z_1, z_2\}$.

Theorem 4.4.1. *Suppose that $n \leq m$, and there exists a permutation σ of the integers $\{1, \dots, 2n\}$, and that*

$$\begin{aligned} dz_1 &= \sum_{1 \leq i \leq n} x^*_{2i-1} \wedge x^*_{2i} \\ dz_2 &= \sum_{1 \leq i \leq n} c_n x^*_{\sigma(2i-1)} \wedge x^*_{\sigma(2i)} \end{aligned}$$

with $\{c_i\}_{1 \leq i \leq n}$ a set of non-zero scalars.

Let G_1 be the graph with vertices labelled by $\{x_i\}_{1 \leq i \leq 2n}$, and set of edges $(x_{2i-1}, x_{2i})_{1 \leq i \leq n}$. Let G_2 be a graph sharing the same vertices, but having edges $(x_{\sigma(2i-1)}, x_{\sigma(2i)})_{1 \leq i \leq n}$.

Then $H^*(L)$ contains a direct sum of 2^c free ΛZ-submodules, where c is the number of distinct 2-colourings of $G_1 \cup G_2$.

Proof. We prove this result by constructing a set of 2^c elements $\alpha_i \in \Lambda^n U^*$ which are independent in $H^*(L)$, such that for all i, α_i and $\underline{\star}\alpha_i$ annihilate dz_1^* and dz_2^*.

First, suppose that linearly independent monomials α_i exist. We immediately see that $d(z_1^* \wedge z_2^* \wedge \alpha_i) = 0$ and $d(z_1^* \wedge z_2^* \wedge \underline{\star}\alpha_i) = 0$ by the Leibnitz rule. From remark 4.1.3, we see that the latter condition is equivalent to $d(\star\alpha_i) = 0$, where $\star$ is the Hodge star on ΛL^*. Now, $d(\star\alpha_i) = 0$ if and only if $\star d(\star\alpha) = \pm\partial\alpha_i = 0$. Since $d\alpha_i = 0$, we see that $\Delta\alpha_i = 0$. Hence $H^*(i_{z_1 \wedge z_2})$ is non-zero, as it maps the class $[z_1^* \wedge z_2^* \wedge \alpha_i]$ to the non-zero class $[\alpha_i]$. By theorem 3.4.5, this shows that $H^*(L)$ contains free ΛZ-submodules generated by $[z_1^* \wedge z_2^* \wedge \alpha_i]$.

Since the α_i are linearly independent monomials in $\ker \Delta \subseteq \Lambda L^*$, and because $\ker \Delta \cong H^*(L)$, they are linearly independent elements of $H^*(L)$.

We construct the α_i as follows.

Note that G_1 and G_2 are both clearly 1-regular graphs. From lemma 2.6.8, we know that G admits 2^c 2-colourings, where c is the number of cycles in G. Denote these 2-colourings $f_i : V(G) \to \{1, 2\}$, with $1 \leq i \leq 2^c$.

Let $A_i = f_i^{-1}(1)$, $B_i = f_i^{-1}(2)$, and define α_i, β_i:

$$\alpha_i = \prod_{x^* \in A_i} x$$
$$\beta_i = \prod_{x^* \in B_i} x$$

Since all vertices in A_i have the same colour, no two share an edge in G. This means that no dz_i^* contains a term $x_i \wedge x_j$ where both x_i^* and x_j^* appear in A_i, for if they did, then G would contain the edge (x_i, x_j), which would contradict f_i being a 2-colouring of G.

Since A_i has n elements, it follows that each term of dz_i^* contains *exactly* one element from A_i. Therefore, α_i annihilates all dz_i^*.

Note that $\underline{\star}\alpha_i$ is a non-zero multiple of β_i, and that β_i also annihilates dz_1^* and dz_2^*.

Thus, $[\alpha_i \wedge z_1^* \wedge z_2^*]$ generate free ΛZ-submodules of $H^*(L)$. □

Example 4.4.2. Let $L = U \oplus Z$ be a nilpotent Lie algebra where

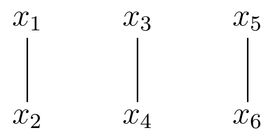
$$U := \text{span}\{x_1, x_2, x_3, x_4, x_5, x_6\},$$
$$Z := \text{span}\{z_1, z_2\},$$

and the differential d for the Koszul complex $(\Lambda L^*, d)$ is given by

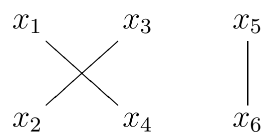
$$dz_1^* = x_1^* \wedge x_2^* + x_3^* \wedge x_4^* + x_5^* \wedge x_6^*$$
$$dz_2^* = x_1^* \wedge x_4^* + x_3^* \wedge x_2^* + x_5^* \wedge x_6^*.$$

The permutation σ has cycle decomposition $(2\,4)$.

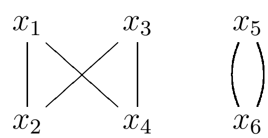
Now the graph G_1, where the edges correspond to summands of dz_1, looks like:



and the graph G_2, where the edges correspond to summands of dz_2, looks like:



Their union is a 2-regular graph consisting of two disjoint cycles:



There are two distinct 2-colourings of the first cycle, $(x_1^* \, x_2^* \, x_3^* \, x_4^*)$:

- Black: x_1^*, x_3^* White: x_2^*, x_4^*
- Black: x_2^*, x_4^* White: x_1^*, x_3^*

There are two distinct 2-colourings of the second cycle, $(x_5^* \, x_6^*)$:

- Black: x_5^* White: x_6^*
- Black: x_6^* White: x_5^*

So we have four distinct 2-colourings of G:

- Black: x_1^*, x_3^*, x_5^* White: x_2^*, x_4^*, x_6^*
- Black: x_2^*, x_4^*, x_5^* White: x_1^*, x_3^*, x_6^*
- Black: x_1^*, x_3^*, x_6^* White: x_2^*, x_4^*, x_5^*
- Black: x_2^*, x_4^*, x_6^* White: x_1^*, x_3^*, x_5^*

Using theorem 4.4.1, we can identify four free ΛZ-submodules of $H^*(L)$, generated by:

- $[x_1^* \wedge x_3^* \wedge x_5^* \wedge z_1^* \wedge z_2^*]$
- $[x_2^* \wedge x_4^* \wedge x_5^* \wedge z_1^* \wedge z_2^*]$
- $[x_1^* \wedge x_3^* \wedge x_6^* \wedge z_1^* \wedge z_2^*]$
- $[x_2^* \wedge x_4^* \wedge x_6^* \wedge z_1^* \wedge z_2^*]$

It is easy to see that their sum is direct.

Example 4.4.3. The nilpotent Lie algebra already explored in example 3.2.3 also satisfies the conditions of theorem 4.4.1. Recall that:

$$\begin{aligned} dz_1^* &= x_1^* \wedge x_2^* + x_3^* \wedge x_4^* \\ dz_2^* &= x_1^* \wedge x_3^* + x_2^* \wedge x_4^* \end{aligned}$$

Here, $\sigma = (2\,3)$.

As we have already seen, the cohomology contains two free ΛZ-submodules, generated by $[x_1^* \wedge x_4^* \wedge z_1^* \wedge z_2^*]$ and $[x_2^* \wedge x_3^* \wedge z_1^* \wedge z_2^*]$.

4.5 The non-triviality of the ΛZ-module structure

The goal of this section is to prove the following result:

Theorem 4.5.1. *Let $L := U \oplus Z$ be a two-step nilpotent Lie algebra with centre Z. Then for every q with $1 \leq q \leq \dim Z^*$, there exists $z \in Z$ and $\omega \in \ker d \cap (\Lambda U^* \otimes \Lambda^q Z^*)$ such that $i_z(\omega) \notin \operatorname{im} d$.*

We begin with a structure result for nilpotent Lie algebras.

Proposition 4.5.2. *Any nilpotent Lie algebra L has an ideal I of codimension 1, and the short exact sequence of Lie algebras*

$$0 \to I \to L \to L/I \to 0$$

gives a sequence of Koszul complexes

$$(\Lambda(L/I)^*, 0) \to (\Lambda(L/I)^* \otimes \Lambda I^*, d) \to (\Lambda I^*, \overline{d}).$$

If we let u be a spanning vector for $(L/I)^*$, then

1. $(\Lambda(L/I)^* \otimes \Lambda I^*, d) \cong (\Lambda L^*, d)$.
2. $d = \overline{d} + u \wedge \theta$ *where* $\theta : \Lambda I^* \to \Lambda I^*$ *is a derivation of degree 0.*
3. $[\theta, \overline{d}] = 0$.

Proof. By theorem 4.2.5, there exists an Engel basis $\{x_i\}_{1 \leq i \leq n}$ for L, such that for i, j satisfying $1 \leq i < j \leq n$, we have $[x_i, x_j] \in \text{span}\{x_k | j+1 \leq k \leq n\}$. Hence, if we let $I := \text{span}\{x_2, \dots, x_n\}$, then $[L, I] \subset I$, and thus I is an ideal of codimension 1.

1. We have
$$\Lambda I^* \otimes \Lambda(L/I)^* = \Lambda(I^* \oplus (L/I)^*) = \Lambda L^*.$$
2. Note that $(\Lambda I^*, \overline{d})$ is the Koszul complex of I, and $\overline{d}^2 = 0$.

   By part (1), an element $x \in \Lambda L^*$ can be written uniquely in the form $x = \alpha + u \wedge \beta$, with $\alpha, \beta \in \Lambda I^*$. Therefore, for any $x \in \Lambda I^*$, $dx = \overline{d}x + u \wedge \beta$, for some unique $\beta \in \Lambda I^*$. We define θ by setting $\theta x := \beta$.

   We note that $\theta x = i_u dx$, and that i_u is zero on ΛI^*. Clearly θ is a map of degree zero. A short calculation shows that θ is a derivation:

$$\begin{aligned}
\theta(a \wedge b) &= i_u d(a \wedge b) \\
&= i_u(da \wedge b + (-1)^{|a|} a \wedge db) \\
&= (i_u da) \wedge b + (-1)^{|a|}(-1)^{|a|} a \wedge (i_u db) \\
&= (i_u da) \wedge b + a \wedge (i_u db) \\
&= \theta a \wedge b + a \wedge \theta b
\end{aligned}$$

3. Finally, we must show that $[\overline{d},\theta]=0$. We have:

$$\begin{aligned} d^2x &= (\overline{d}+u\wedge\theta)(\overline{d}+u\wedge\theta)(x) \\ &= \overline{d}^2(x)+\overline{d}(u\wedge\theta x)+u\wedge\theta\overline{d}x+u\wedge\theta(u\wedge\theta x) \\ &= \overline{d}(u\wedge\theta x)+u\wedge\theta\overline{d}x \\ &= -u\wedge\overline{d}\theta x+u\wedge\theta\overline{d}x \\ &= u\wedge[\theta,\overline{d}] \end{aligned}$$

But $d^2=0$, so $u\wedge[\theta,\overline{d}]$. Since $[\theta,\overline{d}]\in\Lambda I^*$ and multiplication by u is injective on ΛI^*, we conclude that $[\theta,\overline{d}]=0$.

□

The proof of theorem 4.5.1 is by induction on $\dim U$. Using the above proposition, we begin with an ordered basis of U^*, say $\{u_1,\cdots,u_m\}$, and then write the differential d on ΛL^* as follows:

$$d=u_1\wedge\theta_1+\cdots+u_m\wedge\theta_m$$

An increasing filtration of U^*, together with a sequence of differentials, is given by $U_i=\mathrm{span}\{u_1,\ldots,u_i\}$ and $d_i=\sum_{1\le j\le i}u_j\wedge\theta_j$, where $d_i:\Lambda U_i\otimes\Lambda Z^*\to\Lambda U_i\otimes\Lambda Z^*$.

The base case of the induction requires an element $z\in Z$ such that the primary operation

$$i_z:(\Lambda U_0^*\otimes\Lambda^q Z^*,0)\to(\Lambda U_0^*\otimes\Lambda^{q-1}Z^*,0).$$

is non-trivial in cohomology. Recalling example 3.4.7, we observe that since $H^*(\Lambda U_0^*\otimes\Lambda^q Z^*,0)=(\Lambda^q Z^*,0)$, $i:Z\to\mathrm{End}\,(\Lambda Z^*)$ is injective.

At each step of the induction, we produce a non-trivial primary operation

$$i_z:(\Lambda U_k^*\otimes\Lambda^q Z^*,d_k)\to(\Lambda U_k^*\otimes\Lambda^{q-1}Z^*,d_k)$$

We use the following notation in the remainder of the proof:

$$\begin{aligned} \theta &:= \theta_k \\ K &:= \theta(Z^*) \subseteq U^* \\ u &:= u_k \\ \overline{L}^* &:= U_k \oplus Z^* \\ \overline{d} &:= d_{k-1} \\ d &:= d_k \end{aligned}$$

The inductive step begins by assuming the existence of an $\overline{d}$-cycle $\omega \in \Lambda^p U^* \otimes \Lambda^q Z^*$ such that for some $z \in Z$, $i_z(\omega) \notin \operatorname{im} \overline{d}$. The goal here is to produce a new cycle $\omega' \in \Lambda^{p'} U_k^* \otimes \Lambda^q Z^*$ such that $p \leq p' \leq p + \dim Z$ and for some (possibly different) $z' \in Z$, $i_z(\omega') \notin \operatorname{im} d$. The proof of the inductive step proceeds via four lemmas and a nested induction.

Lemma 4.5.3. *If $\alpha \in \Lambda^{\dim K} K$, then $\alpha \wedge \omega \in \ker d$.*

Proof. Note that α annihilates every element of $\theta(\Lambda Z^*)$, and so

$$u \wedge \alpha \wedge \theta\omega = 0$$

Furthermore, since $\alpha \in \Lambda U^*$,

$$\overline{d}(\alpha \wedge \omega) = (-1)^{|\alpha|} \alpha \wedge \overline{d}\omega = 0$$

Hence,

$$d(\alpha \wedge \omega) = \overline{d}(\alpha \wedge \omega) + u \wedge \alpha \wedge \theta\omega = 0$$

This ends the proof of lemma 4.5.3. □

Lemma 4.5.4. *Let $1 \leq l \leq \dim K$. If for all $\gamma \in \Lambda^l K$ and for all $z \in Z$, $i_z(\gamma \wedge \omega) \in \operatorname{im} \overline{d}$, then for every $\alpha \in \Lambda^{l-1} K$, there exists $\beta \in \Lambda^{p+l-1} U^* \otimes \Lambda^q Z^*$ such that $d(\alpha \wedge \omega + u \wedge \beta) = 0$.*

Proof. Let $\alpha \in \Lambda^{l-1}K$. Since $\overline{d}(\alpha \wedge \omega) = 0$,

$$\begin{aligned} d(\alpha \wedge \omega) &= u \wedge \theta(\alpha \wedge \omega) \\ &= u \wedge \sum_j \theta z_j^* \wedge i_{z_j}(\alpha \wedge \omega) \\ &= -u \wedge \sum_j i_{z_j}(\theta z_j^* \wedge \alpha \wedge \omega) \end{aligned}$$

Now, for all j, $\theta z_j^* \wedge \alpha \in \Lambda^l K$. Our assumption implies that $i_{z_j}(\theta z_j^* \wedge \alpha \wedge \omega) \in \operatorname{im} \overline{d}$. Therefore for each j, there exists $\beta_j \in \Lambda^{p+l-1}U^* \otimes \Lambda^q Z^*$ satisfying

$$\overline{d}\beta_j = i_{z_j}(\theta z_j^* \wedge \alpha \wedge \omega)$$

We define $\beta := \sum_j \beta_j$. Note that $\beta \in \Lambda^{p+l-1}U^* \otimes \Lambda^q Z^*$ and that

$$d(u \wedge \beta) = -u \wedge \overline{d}\beta = u \wedge \alpha \wedge \theta\omega$$

Now a simple calculation shows that $\alpha \wedge \omega + u \wedge \beta$ is a d-cycle, as required. This ends the proof of lemma 4.5.4. □

Lemma 4.5.5. *If $x \in \Lambda\overline{L}^* \subset \Lambda L^*$ and $x \notin \operatorname{im}\overline{d}$, then $x + u \wedge y \notin \operatorname{im} d$ for any $y \in \Lambda\overline{L}^*$.*

Proof. Suppose that for some $a, b, y \in \Lambda\overline{L}^*$, $d(a + u \wedge b) = x + u \wedge y$. Then,

$$d(a + u \wedge b) = \overline{d}a + u \wedge \theta a + u \wedge \overline{d}b = x + u \wedge y$$

Since $i_u(x) = 0$ (intuitively, x does not have a summand which is a multiple of u), it follows that $x = \overline{d}a$ and $y = \theta a + \overline{d}b$. But this contradicts the assumption that $x \notin \operatorname{im}\overline{d}$. Therefore $x + u \wedge y \notin \operatorname{im} d$. This ends the proof of lemma 4.5.5. □

Lemma 4.5.6. *Suppose $0 \leq l \leq \dim K$, and that for each $\alpha \in \Lambda^l K$, there exists β such that $d(\alpha \wedge \omega + u \wedge \beta) = 0$. Then either:*

1. *there exists $z \in Z$ such that*

$$i_z(\alpha \wedge \omega + u \wedge \beta) \notin \operatorname{im} d$$

(this is always the case if $l = 0$)

2. *or, the hypotheses of lemma 4.5.4 are satisfied.*

Proof. We consider the two cases:

1. If for some $\alpha \in \Lambda^l K$, there exists $z \in Z$ with $i_z(\alpha \wedge \omega) \notin \operatorname{im} \overline{d}$, then lemma 4.5.5 forces
$$i_z(\alpha \wedge \omega + u \wedge \beta) = i_z(\alpha \wedge \omega) - u \wedge i_z \beta \notin \operatorname{im} d$$
and so i_z maps the non-trivial cycle $\alpha \wedge \omega + u \wedge \beta$ to a non-trivial cycle, thus $H^*(i_z) \neq 0$ and we are done.

   If $l = 0$, then $\alpha \in \mathbb{R}$, so $i_z(\alpha\omega) = \alpha i_z(\omega) \notin \operatorname{im} \overline{d}$ by the inductive hypothesis.

2. The only other possibility now is that for all $\alpha \in \Lambda^l K$ and for all $z \in Z$, $i_z(\alpha \wedge \omega) \in \operatorname{im} \overline{d}$. In this case, the hypotheses of lemma 4.5.4 are satisfied.

This ends the proof of lemma 4.5.6. □

Using the above results, we are able to proceed with the inductive step itself, namely that there exists $\alpha \in \Lambda K$ satisfying the following two properties:

- For some $\beta \in \Lambda^{p+|\alpha|-1} U^* \otimes \Lambda^q Z^*$, $\alpha \wedge \omega + u \wedge \beta \in \ker d$
- For some $z \in Z$, $i_z(\alpha \wedge \omega + u \wedge \beta) \notin \operatorname{im} d$

We begin by looking for α in $\Lambda^{\dim K} K$. Lemma 4.5.3 gives a d-cycle $\alpha \wedge \omega$ for $\alpha \in \Lambda^{\dim K} K$. We apply lemma 4.5.6 with $l = \dim K$ and $\beta = 0$. We either get a non-trivial primary operation in which case we are done, or we are able to apply lemma 4.5.4 to reduce l by one, and apply lemma 4.5.6 again. This process eventually terminates because lemma 4.5.6 always yields a non-trivial primary operation if the hypotheses are satisfied and $l = 0$.

This closes the induction and completes the proof of theorem 4.5.1.

We have thus established conjecture 3.4.6 in the case where L is a two-step nilpotent Lie algebra. We also get an immediate corollary.

Corollary 4.5.7. *For every q with $0 \leq q \leq \dim Z$, there exists a non-trivial cycle in $(\Lambda U^* \otimes \Lambda^q Z^*, d)$. Furthermore, if $q < \dim Z$, this cycle is in the image of a primary operation.*

Proof. We consider two cases:

- For $q > 0$, the theorem yields a non-trivial cycle $\omega \in \Lambda U^* \otimes \Lambda^q Z^*$.
- For $q = 0$, use the theorem to produce a non-trivial cycle $\omega \in \Lambda U^* \otimes Z^*$ which satisfies $i_z(\omega) \notin \operatorname{im} d$ for some z. Now $i_z(\omega) \in \Lambda U^*$ is a non-trivial cycle.

□

Example 4.5.8. We will use this theorem to show that the cohomology of the Lie algebra of example 3.2.3 has a non-trivial ΛZ-module structure.

Let $L := U \oplus Z$, where $U = \operatorname{span}\{x_1, x_2, x_3, x_4\}$ and $Z = \{z_1, z_2\}$, with

$$\begin{aligned} [x_1, x_2] &:= z_1 \\ [x_3, x_4] &:= z_1 \\ [x_1, x_3] &:= z_2 \\ [x_2, x_4] &:= z_2 \end{aligned}$$

and all other brackets zero.

Under the corresponding dual basis, the differential is defined as follows:

$$\begin{aligned} dz_1^* &= x_1^* \wedge x_2^* + x_3^* \wedge x_4^* \\ dz_2^* &= x_1^* \wedge x_3^* + x_2^* \wedge x_4^* \end{aligned}$$

$(\Lambda U_0^* \otimes \Lambda Z^*, d_0)$ We can take $H^*(i_{z_1})$ as our operation, since $d_0 = 0$ and $H^*(i_{z_1})(z_1^*) = 1 \neq 0$.

$(\Lambda U_1^* \otimes \Lambda Z^*, d_1)$ Since $U_1 = \operatorname{span}\{x_1^*\}$, we have that $\theta_1 = 0$, so again we can take $H^*(i_{z_1})$ as our operation, because $H^*(i_z)(z_1^*) = 1 \neq 0$.

$(\Lambda U_2^* \otimes \Lambda Z^*, d_2)$ Let $u := x_2^*$. We make the following calculations:

$$\begin{aligned} U_2 &= \operatorname{span}\{x_1^*, x_2^*\} \\ \theta_2 z_1^* &= -x_1^* \\ \theta_2 z_2^* &= 0 \\ K &= \operatorname{span}\{x_1^*\} \end{aligned}$$

Now x_1^* is a suitable orientation of ΛK, and by lemma 4.5.3, $x_1^* \wedge z_1^*$ is a cycle. This is a non-zero cycle, since $H^*(i_{z_1})(x_1^* \wedge z_1^*) = -x_1^* \notin \operatorname{im} d$.

$(\Lambda U_3^* \otimes \Lambda Z^*, d_3)$ Let $u := x_3^*$. We make the following calculations:

$$U_3 = \operatorname{span}\{x_1^*, x_2^*, x_3^*\}$$
$$\theta_3 z_1^* = 0$$
$$\theta_3 z_2^* = -x_1^*$$
$$K = \operatorname{span}\{x_1^*\}$$

Again, x_1^* is a suitable orientation of ΛK. Since $x_1^* \wedge x_1^* \wedge z_1^* = 0$, lemma 4.5.4 tells us that $x_1^* \wedge z_1^*$ is a cycle in $(\Lambda U_3 \otimes \Lambda Z, d_3)$, and we are done.

$(\Lambda U_4^* \otimes \Lambda Z^*, d_4)$ Let $u := x_4^*$. We make the following calculations:

$$U_4 = \operatorname{span}\{x_1^*, x_2^*, x_3, x_4\}$$
$$\theta_4 z_1^* = -x_3^*$$
$$\theta_4 z_2^* = -x_2^*$$
$$K = \operatorname{span}\{x_2^*, x_3^*\}$$

We pick $x_2^* \wedge x_3^*$ as an orientation of ΛK. Note the following:

$$\begin{aligned} i_{z_1}(x_2^* \wedge x_3^* \wedge x_1^* \wedge z_1^*) &= -x_2^* \wedge x_3^* \wedge x_1^* \\ &= d(x_1^* \wedge z_1^*) \\ i_{z_2}(x_2^* \wedge x_3^* \wedge x_1^* \wedge z_1^*) &= 0 \end{aligned}$$

Lemma 4.5.6 implies that for all $k \in K$, $\theta_4(k \wedge x_1^* \wedge z_1^*) \in \operatorname{im} d_4$. We compute:

$$\theta_4(x_2^* \wedge x_1^* \wedge z_1^*) = -x_2^* \wedge x_1^* \wedge x_3^* = d_4(x_2^* \wedge z_2^*)$$
$$\theta_4(x_3^* \wedge x_1^* \wedge z_1^*) = 0$$

So we have:

$$\begin{aligned} d_4(x_2^* \wedge x_1^* \wedge z_1^*) &= d_3(x_2^* \wedge x_1^* \wedge z_1^*) + x_4^* \wedge \theta(x_2^* \wedge x_1^* \wedge z_1^*) \\ &= x_4^* \wedge d_4(x_2^* \wedge z_2^*) \\ &= d_4(-x_4^* \wedge x_2^* \wedge z_2^*) \end{aligned}$$

Therefore:

$$d_4(x_2^* \wedge x_1^* \wedge z_1^* - x_4^* \wedge x_2^* \wedge z_2^*) = 0$$

We are done, since $H^*(i_{z_1})(x_2^* \wedge x_1^* \wedge z_1^* - x_4^* \wedge x_2^* \wedge z_2^*) = x_2^* \wedge x_1^* \notin \operatorname{im} d_4$, thus $H^*(i_{z_1}) \neq 0$.

Chapter 5

Open problems

We hope this thesis has inspired the reader to look further and investigate the following problems:

- Cairns and Jessup conjectured that theorem 4.5.1 holds for all nilpotent Lie algebras. While we have answered the conjecture affirmatively for two-steps, this problem is still open for Lie algebras of nilpotency degree greater than 2.

- In section 4.2, it was shown that if F_n is a free two-step nilpotent Lie algebra with centre Z, then for all non-zero $z \in Z$, $H^*(i_z) \neq 0$. Section 4.5 shows that for any nilpotent Lie algebra L, there exists some non-zero $z \in Z$ with $H^*(i_z) \neq 0$. All calculations performed to date lead us to believe that the stronger form of the statement which is true for free two-steps actually holds for all two-step nilpotent Lie algebras, however we were unable to prove or disprove this fact.

- While free two-steps do not have free ΛZ-submodules, there is no free two-step with centre of dimension two. The results in sections 4.3 and 4.4 seem to suggest that perhaps *all* two-steps with centre of dimension two have cohomology spaces with free ΛZ-submodules.

- The Toral rank conjecture (1.0.1) is still open.

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